SUBJECT CODE : 204191

Choice Based Credit System

SAVITRIBAI PHULE PUNE UNIVERSITY - 2019 SYLLABUS

S.E. (E & TC / Electronics) Semester - IV

SIGNALS & SYSTEMS

(For END SEM Exam - 70 Marks)

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FEATURES

- ☑ Written by Popular Authors of Text Books of Technical Publications
- ☑ Covers Entire Syllabus
- ☑ Question - Answer Format
- ☑ Exact Answers and Solutions
- ☑ Important Points to Remember
- ☑ Formulae at a Glance
- ☑ Chapterwise Solved SPPU Questions Dec.-2003 to Dec.-2022

SOLVED SPPU QUESTION PAPERS

- May - 2017
- Dec. - 2017
- May - 2018
- Dec. - 2018
- May - 2019
- Dec. - 2019
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- Dec. - 2022

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A Guide For Engineering Students

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3 Fourier Series

3.1 : Fourier Series Representation of Periodic CT Signals

Important Points to Remember

- Fourier representation provides time frequency conversion of signals.
- The signal satisfies Dirichlet conditions if its Fourier series is convergent.

Q.1 State Dirichlet conditions for existance of fourier series.

[SPPU : May-19, Marks 5]

Ans. : i) Single valued property : $x(t)$ must have only one value at any time instant within the interval T_0.

ii) Finite discontinuities : $x(t)$ should have at the most finite number of discontinuities in the interval T_0. Because of this, the signal can be represented mathematically.

iii) Finite peaks : The signal $x(t)$ should have finite number of maxima and minima in the interval T_0.

iv) Absolute integrability : The signal $x(t)$ should be absolutely integrable, i.e. $\int_{<T_0>} |x(t)| < \infty$. This is because the analysis equation integrates $x(t)$.

- Above conditions are sufficient but not necessary conditions for Fourier series representation.
- Most of physical signal satisfy above conditions.

Q.2 State the necessity of Fourier representations.

Ans. : Necessity of fourier representations :

Fourier representations are necessary because of following reasons :

1. All the signals cannot be fully analyzed in time domain. They can be analyzed in frequency domain also. Fourier representations provide the time-frequency conversion of signals.
2. The signals need to be represented in terms of orthogonal basis functions. This is done with the help of Fourier representation.
3. The frequency domain analysis of the LTI systems such as pole-zero plots, frequency response etc. can be obtained with Fourier representations.
4. The response of LTI system to any deterministic signal can be obtained with the help of Fourier representation.

3.2 : Orthogonality

Important Points to Remember

- Fourier series is used to represent any periodic signal in terms of complex exponentials. These complex exponentials are called basis functions.
- The vectors 'f' and 'x' are said to be orthogonal if their dot product is zero. In other words, vectors are orthogonal if they are mutually perpendicular. i.e.,

$$f \cdot x = |f|\,|x| \cos\theta = 0$$

- Orthogonality of real set of signals $x_1(t), x_2(t), x_3(t) \ldots x_N(t)$ over an interval (t_1, t_2) is given as,

$$\int_{t_1}^{t_2} x_m(t) x_n(t)\, dt = \begin{cases} 0 & for\, m \neq n \\ E_n & for\, m = n \end{cases}$$

Here 'E_n' is an energy of the signal x (t). When $E_n = 1$ for all values of 'n', then it is called orthonormal set.

Q.3 Show that the following signals are orthogonal over an interval $[0,1]$.

$f(t) = 1$ **and** $x(t) = \sqrt{3}\,(1-2t)$

Ans. : The signals are orthogonal if,

$$\int_{t_1}^{t_2} f(t)x(t)\, dt = 0$$

$$\int_{t_1}^{t_2} f(t)x(t)\, dt = \int_0^1 1\left[\sqrt{3}(1-2t)\right] dt$$

$$= \int_0^1 \sqrt{3}\, dt - \int_0^1 2\sqrt{3}\, t\, dt$$

$$= \sqrt{3}\int_0^1 dt - 2\sqrt{3}\int_0^1 t\, dt$$

$$= \sqrt{3}\,[t]_0^1 - 2\sqrt{3}\left[\frac{t^2}{2}\right]_0^1 = 0$$

Thus the two given signals are orthogonal over an interval $[0,1]$.

Q.4 Show that over the period of interval '0' to 2π, a rectangular function is orthogonal to signals *$\cos t$, $\cos 2t$, ... $\cos nt$* for all integer values of n.

Ans. : The rectangular function over the period 0 to 2π can be expressed as,

$$f(t) = \begin{cases} 1 & for \quad 0 \le t \le 2\pi \\ 0 & otherwise \end{cases}$$

Here $\quad x(t) = \cos nt$

Let us check orthogonality of $f(t)$ and $x(t)$ i.e.,

$$\int_t^{t+T_0} f(t)\, x(t)\, dt = \int_t^{t+T_0} 1 \cos nt\, dt$$

$$= \int_0^{2\pi} \cos nt\, dt$$

$$= \left[\frac{\sin nt}{n}\right]_0^{2\pi}$$

$$= \frac{1}{n}\left[\sin 2\pi t - \sin 0\right] = 0$$

Thus $\cos nt$ and rectangular function are orthogonal over an interval 0 to 2π.

3.3 : Fourier Series (FS) Representation of CT Signals

Important Points to Remember

- Fourier series is used to represent periodic CT signals.
- The fourier series is of three types : Trigonometric, compact trigonometric and exponential fourier series.

Q.5 Define the three types of fourier series.

[SPPU : May-19, Marks 3, June-22, Dec.-22, Marks 6]

Ans. : 1) Trigonometric fourier series

It is given as,

$$x(t) = a(0) + \sum_{k=1}^{\infty} a(k)\cos k\omega_0 t + \sum_{k=1}^{\infty} b(k)\sin k\omega_0 t$$

where $a(0) = \frac{1}{T}\int_{<T>} x(t)dt$

$$a(k) = \frac{2}{T}\int_{<T>} x(t)\cos k\omega_0 t\, dt \quad \ldots(Q.5.1)$$

$$b(k) = \frac{2}{T}\int_{<T>} x(t)\sin k\omega_0 t\, dt$$

- Here $\int_{<T>}$ indicates integration over one time period.
- And $\omega_0 = \frac{2\pi}{T}$, where 'T' is period of the signal *x(t)*. This form of Fourier series is also called quadrature Fourier series.

2) Compact Trigonometric Fourier Series

It is also called polar fourier series. It is given as follows :

$$x(t) = D(0) + \sum_{k=1}^{\infty} D(k)\cos(k\omega_0 t + \phi(k))$$

where $D(0) = a_0 = \frac{1}{T}\int_{<T>} x(t)dt \quad \ldots(Q.5.2)$

$$D(k) = \sqrt{a(k)^2 + b(k)^2} \text{ and } \phi(k) = -\tan^{-1}\left(\frac{b(k)}{a(k)}\right)$$

3) Exponential Fourier Series

It is given as,

$$x(t) = \sum_{k=-\infty}^{\infty} X(k)\, e^{jk\omega_0 t} \quad \text{(synthesis equation)}$$

where $X(k) = \frac{1}{T}\int_{<T>} e^{-jk\omega_0 t}dt$ (analysis equation) $\ldots(Q.5.3)$

- Here *X(k)* are called Fourier Series coefficients.
- *x(t)* and *X(k)* are represented by the Fourier Series (FS) pair as,

$$x(t) \xleftrightarrow{FS} X(k)$$

Q.6 State the applications of Fourier series.

Ans. : i) Harmonic analysis of periodic signals.

ii) Advanced noise cancellation in digital filtering.

iii) Trigonometric Fourier series is used in approximation theory.

iv) Control theory uses Fourier series.

v) Vibration, acoustical analysis and electrical engineering.

vi) Optics, image processing and quantum mechanics.

vii) Solution of differential equation.

viii) Solution of Basel problem by Parseval's theorem.

3.4 : Properties of Fourier Series and Their Physical Significance

Important Points to Remember

- Fourier series exhibits many properties which are useful for quick analysis of signals.

Convolution : $x(t) * y(t) \xleftrightarrow{FS} TX(k) \cdot Y(k)$

Time shift : $x(t-t_0) \xleftrightarrow{FS} e^{-jk\omega_0 t_0} X(k)$

Frequency shift : $e^{jk_0\omega_0 t} \xleftrightarrow{FS} X(k-k_0)$

Modulation : $x(t) \cdot y(t) \xleftrightarrow{FS} X(k) * Y(k)$

Parseval's theorem : Power, $P = \sum_{k=-\infty}^{\infty} |X(k)|^2$

Q.7 State and prove following properties of CTFS.
i) Differentiation in time ii) convolution in time. [Dec.-16, Marks 4]

Ans. : i) Time Differentiation :

$$\boxed{\frac{dx(t)}{dt} \xleftrightarrow{FS} jk\omega_0 X(k)} \quad \dots (Q.7.1)$$

Proof : $x(t) = \sum_{k=-\infty}^{\infty} X(k) e^{jk\omega_0 t}$ By definition of FS ... (Q.7.2)

Differentiating with respect to 't',

$$\frac{dx(t)}{dt} = \sum_{k=-\infty}^{\infty} X(k)\, jk\omega_0\, e^{jk\omega_0 t}$$

$$\therefore \quad \frac{dx(t)}{dt} = \sum_{k=-\infty}^{\infty} [jk\omega_0 X(k)] e^{jk\omega_0 t}$$

Comparing above equation with equation (Q.7.2) we get,

$$\frac{dx(t)}{dt} \xleftrightarrow{FS} jk\omega_0 X(k)$$

ii) Convolution in Time

$$\boxed{z(t) = x(t) * y(t) \xleftrightarrow{FS} Z(k) = T\,X(k)Y(k)} \quad \dots (Q.7.3)$$

Proof : $Z(k) = \frac{1}{T} \int_{<T>} z(t) e^{-jk\omega_0 t} dt = \frac{1}{T} \int_{<T>} [x(t) * y(t)] e^{-jk\omega_0 t} dt$ ...(Q.7.4)

$x(t) * y(t) = \int_{<T>} x(\tau) y(t-\tau) d\tau$. This convolution is performed over one period for periodic signals. Putting this convolution in equation (Q.7.4) we get,

$$Z(k) = \frac{1}{T} \int_{<T>} \int_{<T>} x(\tau) y(t-\tau) d\tau\, e^{-jk\omega_0 t} d\tau\, dt$$

Interchanging the order of integrations,

$$Z(k) = \frac{1}{T} \int_{<T>} x(\tau) \int_{<T>} y(t-\tau) e^{-jk\omega_0 t} d\tau\, dt$$

Put $t - \tau = m$. Therefore $dt = dm$. Since integration is over one period, this substitution will just shift the integrating limits. But it will be again over one period only. Hence we can write,

$$Z(k) = \frac{1}{T} \int_{<T>} x(\tau) \int_{<T>} y(m) e^{-jk\omega_0 (\tau+m)} d\tau\, dm$$

$$= \frac{1}{T} \int_{<T>} x(\tau) \int_{<T>} y(m) e^{-jk\omega_0 \tau} \cdot e^{-jk\omega_0 m} d\tau\, dm$$

$$= \frac{1}{T} \int_{<T>} x(\tau) e^{-jk\omega_0 \tau} d\tau \int_{<T>} y(m) e^{-jk\omega_0 m} dm$$

$$= \frac{1}{T} [T\,X(k)] \cdot [T\,Y(k)] = T\,X(k)\,Y(k)$$

Significance : Convolution of two periodic signals results in multiplication of their Fourier coefficients and period T.

Q.8 Explain Parseval's theorem for fourier series.
[SPPU : Dec.-16, June-22, Marks 2]

Ans. : If $x(t)$ is the periodic power signal with Fourier coefficients $X(k)$, then average power in the signal is given by $\sum_{k=-\infty}^{\infty} |X(k)|^2$, i.e.,

$$\text{Power,} \quad P = \sum_{k=-\infty}^{\infty} |X(k)|^2 \qquad \dots (Q.8.1)$$

Proof : The power in the signal $x(t)$ is given as,

$$P = \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 \, dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t)x^*(t)\,dt \qquad \dots (Q.8.2)$$

We have, $x(t) = \sum_{k=-\infty}^{\infty} X(k)e^{jk\omega_0 t}$ by synthesis equation

$$\therefore x^*(t) = \left[\sum_{k=-\infty}^{\infty} X(k)e^{jk\omega_0 t}\right]^* \text{ by taking conjugates of both sides}$$

$$= \sum_{k=-\infty}^{\infty} X^*(k)e^{-jk\omega_0 t}$$

Putting above expression of $x^*(t)$ in equation (Q.8.2),

$$P = \frac{1}{T}\int_{-T/2}^{T/2} x(t) \sum_{k=-\infty}^{\infty} X^*(k)e^{-jk\omega_0 t}\, dt$$

Here $\int_{-T/2}^{T/2} = \int_{<T>}$ i.e. integration over one period of $x(t)$. Interchanging the order of summation and integration,

$$P = \sum_{k=-\infty}^{\infty} X^*(k)\cdot\frac{1}{T}\int_{<T>} x(t)e^{-jk\omega_0 t}\,dt = \sum_{k=-\infty}^{\infty} X^*(k)X(k) = \sum_{k=-\infty}^{\infty} |X(k)|^2$$

Significance : Power of the signal can be obtained by squaring and adding the magnitudes of Fourier coefficients.

Q.9 Explain Gibbs phenomenon.

☞ [SPPU : Dec.-19, Marks 3, June-22, Dec.-22 Marks 4]

Ans. : Consider the Fourier series for square wave, i.e.,

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{A}{k\pi}\sin\left(\frac{k\omega_0\tau}{2}\right)e^{jk\omega_0 t}$$

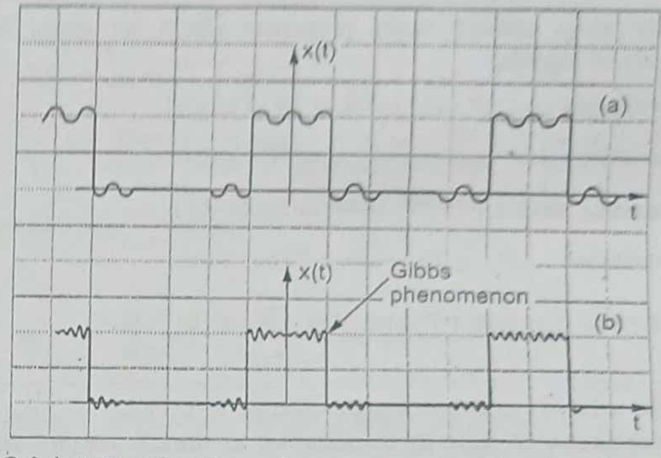


Fig. Q.9.1 Approximation of square wave using Fourier series

- Above equation represents square wave in terms of infinite number of sinsoidal terms. When limited number of terms are taken, then the square wave is approximated as shown in Fig. Q.9.1.
- There is overshoot and ripple near discontinuities. It goes on reducing as number of terms are increased in approximation of waveform.
- Fig. Q.9.1 (a) uses less number of terms compared to Fig. Q.9.1 (b). As the number of terms in Fourier representation becomes very high, then the waveform will appear just like original square wave.
- This overshoot near discontinuity is called Gibbs phehomenon. As the number of terms of Fourier series representation increase, the error between original and approximated signal contains no signal energy. Then the waveform appears close to original.

Q.10 Find the quadrature fourier series for the full wave rectifier output signal. With amplitude 'A' and period 0 to π.

☞ [SPPU : May-14, Marks 6]

Ans. : Here $x(t) = A \sin \omega_0 t$

Since half cycle is from 0 to π. $\omega_0 = 1$ (i.e one cycle of 2π radians).

$\therefore \quad x(t) = A \sin t$

Take $T = \pi$ since this is full wave rectified signal.

$$\therefore \quad a(0) = \frac{1}{T} \int_{<T>} A \sin t \, dt = \frac{2A}{\pi}$$

$$a(k) = \frac{2}{T} \int_{<T>} x(t) \cos k\omega_0 t dt = \frac{2}{\pi} \int_0^{\pi} A \sin t \cos kt \, dt$$

$$= \frac{2A}{\pi} \int_0^{\pi} \frac{1}{2} [\sin(t - kt) + \sin(t + kt)] dt$$

$$= \frac{A}{\pi} \left[\int_0^{\pi} \sin(1-k)t \, dt + \int_0^{\pi} \sin(1+k)t \, dt \right]$$

$$= \frac{A}{\pi} \left[\frac{-\cos(1-k)t}{(1-k)} + \frac{-\cos(1+k)t}{1+k} \right]_0^{\pi}$$

$$= \frac{A}{\pi} \left[\frac{-\cos(1-k)\pi + \cos 0}{1-k} + \frac{-\cos(1+k)\pi + \cos 0}{1+k} \right]$$

$$= \frac{A}{\pi} \left[\frac{1-\cos(1-k)\pi}{1-k} + \frac{1-\cos(1+k)\pi}{1+k} \right]$$

Here $\left.\begin{matrix} 1-\cos(1-k)\pi \\ 1-\cos(1+k)\pi \end{matrix}\right\} = \begin{cases} 0 & \text{for } k = 1, 3, 5, 7 \ldots.. \\ 2 & \text{for } k = 0, 2, 4, 6 \ldots.. \end{cases}$

$$\therefore \quad a(k) = \frac{A}{\pi} \left[\frac{2}{1-k} + \frac{2}{1+k} \right] \text{ for } k = 0, 2, 4, 6$$

$$= \begin{cases} \dfrac{4A}{\pi(1-k^2)} & \text{for } k = 0, 2, 4, 6 \ldots.. \\ 0 & \text{for } k = 1, 3, 5, 7 \ldots.. \end{cases}$$

$$b(k) = \frac{2}{T} \int_{<T>} x(t) \cdot \sin k\omega_0 t \, dt = \frac{2}{\pi} \int_0^{\pi} A \sin t \cdot \sin kt \, dt$$

$$= \frac{2A}{\pi} \int_0^{\pi} \frac{1}{2} [\cos(t - kt) - \cos(t + kt)] dt$$

$$= \frac{A}{\pi} \left[\int_0^{\pi} \cos(1-k)t \, dt - \int_0^{\pi} \cos(1+k)t \, dt \right]$$

$$= \frac{A}{\pi} \left[\frac{\sin(1-k)t}{1-k} - \frac{\sin(1+k)t}{(1+k)} \right]_0^{\pi} = 0 \text{ for all } k.$$

Thus the quadrature fourier series will be,

$$x(t) = a(0) + \sum_{k=1}^{\infty} a(k) \cos k\omega_0 t + \sum_{k=1}^{\infty} b(k) \sin k\omega_0 t$$

$$= \frac{2A}{\pi} + \sum_{k=2,4,6\ldots} \frac{4A}{\pi(1-k^2)} \cos kt$$

Q.11 Find the trigonometric Fourier series for the periodic signal x(t) shown in the following figure and sketch the amplitude and phase spectra : [SPPU : Dec.-14,15,18,19, May-15,19, Marks 6]

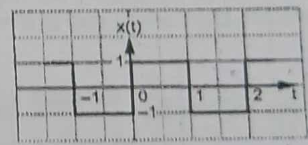


Fig. Q.11.1

Ans. :

$$a(0) = \frac{1}{T} \int_{<T>} x(t) \, dt = \frac{1}{2} \left[\int_0^1 dt + \int_1^2 -1 \, dt \right] = 0$$

$$a(k) = \frac{2}{T} \int_{<T>} x(t) \cos k\omega_0 t \, dt$$

$$= \frac{2}{2} \left[\int_0^1 \cos k\omega_0 t \, dt + \int_1^2 -\cos k\omega_0 t \, dt \right]$$

$$= \int_0^1 \cos k\pi t\, dt - \int_1^2 \cos k\pi t\, dt \text{ , Here } \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$$

$$= \left.\frac{\sin k\pi t}{k\pi}\right|_0^1 - \left.\frac{\sin k\pi}{k\pi}\right|_1^2$$

$$= \frac{1}{k\pi}\{\sin k\pi - \sin 0 - [\sin 2k\pi - \sin k\pi]\} = 0,$$

since $\sin nk\pi = 0$

$$b(k) = \frac{2}{T}\int_{<T>} x(t)\sin k\omega_0 t\, dt = \frac{2}{2}\left[\int_0^1 \sin k\omega_0 t\, dt + \int_1^2 -\sin k\omega_0 t\, dt\right]$$

$$= \int_0^1 \sin k\pi t\, dt - \int_1^2 \sin k\pi t\, dt \text{ , since } \omega_0 = \pi$$

$$= \left.\frac{-\cos k\pi t}{k\pi}\right|_0^1 - \left.\frac{-\cos k\pi t}{k\pi}\right|_1^2$$

$$= \frac{1}{k\pi}\{-\cos k\pi + \cos 0 + \cos 2k\pi - \cos k\pi\}$$

$$= \frac{1}{k\pi}[2 - 2\cos k\pi] \text{ , since } \cos 2k\pi = 1 \text{ for all } k$$

$$= \frac{2}{k\pi}[1 - \cos k\pi] = \begin{cases} \frac{4}{k\pi} & \text{for odd } k \\ 0 & \text{for even } k \end{cases}$$

Trigonomatric Fourier series is given as,

$$x(t) = a(0) + \sum_{k=1}^{\infty} a(k)\cos k\omega_0 t + \sum_{k=1}^{\infty} b(k)\sin k\omega_0 t$$

$$= \sum_{k=1,3,5}^{\infty} \frac{4}{k\pi}\sin k\omega_0 t$$

Magnitude and phase plot

$$|c(k)| = \sqrt{a^2(k) + b^2(k)} = b(k) \text{ since } a(k) = 0$$

$$\therefore \quad |c(k)| = \frac{4}{k\pi} \text{ for odd '}k\text{'}$$

$$\angle c(k) = \tan^{-1}\frac{b(k)}{a(k)} = \tan^{-1}\frac{b(k)}{0} = \frac{\pi}{2}$$

k	1	3	5	7
$c(k)$	$4/\pi$	$4/3\pi$	$4/5\pi$	$4/7\pi$
$\angle c(k)$	$\pi/2$	$\pi/2$	$\pi/2$	$\pi/2$

Fig. Q.11.2 shows magintude spectra. Note that phase is constant at $\pi/2$ for all 'k'.

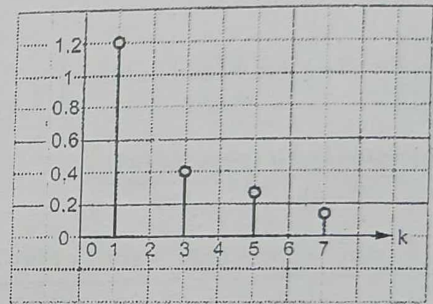


Fig. Q.11.2 Magnitude spectra

Q.12 Determine the trigonometric Fourier series of the signal shown in Fig. Q.12.1. [SPPU : May-16, Dec.-17, Marks 6, Dec.-22, Marks 8]

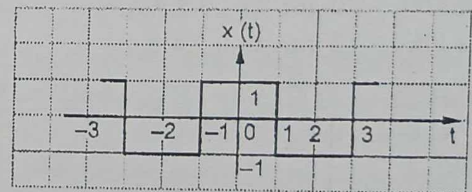


Fig. Q.12.1 Square wave

Ans. : This waveform has even and half wave symmetry. Hence $b(k) = 0$ for all k and $a(0) = 0$.

$$b(k) = \frac{4}{\tau}\int_{(T/2)} x(t)\cos k\omega_o t\, dt$$

Here $\omega_o = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$

$$\therefore \quad b(k) = \frac{4}{4}\left[\int_0^1 \cos k\frac{\pi}{2}t\,dt + \int_1^2 (-1)\cos k\frac{\pi}{2}t\,dt\right]$$

$$= \left[\frac{\sin k\frac{\pi}{2}t}{k\frac{\pi}{2}}\right]_0^1 - \left[\frac{\sin k\frac{\pi}{2}t}{k\frac{\pi}{2}}\right]_1^2$$

$$= \frac{2}{k\pi}\left[\sin\frac{k\pi}{2} - \sin 0 - \sin k\pi + \sin\frac{k\pi}{2}\right]$$

$$= \frac{4}{k\pi}\sin\frac{k\pi}{2} \quad \text{Here } \sin\frac{k\pi}{2} = 0 \text{ for even 'k'}$$

$$= \begin{cases} \frac{4}{k\pi} \sin\frac{k\pi}{2} & \text{for odd } k \\ 0 & \text{for even } k \end{cases}$$

$$= \begin{cases} \pm\frac{4}{k\pi} & \text{for odd } k \\ 0 & \text{for even } k \end{cases}$$

Trigonometric Fourier series is given as,

$$x(t) = a(o) + \sum_{k=1}^{\infty} a(k)\cos k\omega_o t + \sum_{k=1}^{\infty} b(k)\sin k\omega_o t$$

$$= \sum_{k=1,3,5}^{\infty} \pm\frac{4}{k\pi}\sin k\omega_o t$$

Q.13 Determine the complex exponential Fourier series for periodic rectangular pulse train shown in Fig. Q.13.1.

[SPPU : Dec.-04, 05, May-18, Marks 10]

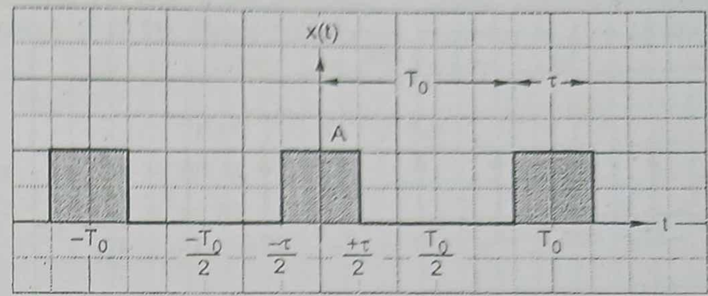


Fig. Q.13.1 Periodic rectangular pulse train

Ans. : Step 1 : To obtain $X(k)$

$$X(k) = \frac{1}{T}\int_{<T>} x(t)e^{-jk\omega_0 t}dt \quad \text{... By definition of FS}$$

$$= \frac{1}{T_0}\int_{-\tau/2}^{\tau/2} A\,e^{-jk\omega_0 t}\,dt = \frac{A}{T_0}\cdot\frac{1}{-jk\omega_0}\left[e^{-jk\omega_0 t}\right]_{-\tau/2}^{\tau/2}$$

$$= \frac{A}{T_0}\cdot\frac{1}{-jk\cdot\frac{2\pi}{T_0}}\left[e^{-jk\omega_0\tau/2} - e^{jk\omega_0\tau/2}\right], \text{ since } \omega_0 = \frac{2\pi}{T_0}$$

$$= \frac{A}{k\pi}\left\{\frac{e^{jk\omega_0\tau/2} - e^{-jk\omega_0\tau/2}}{2j}\right\} \text{ Rearranging the equation}$$

$$= \frac{A}{k\pi}\sin\left(\frac{k\omega_0\tau}{2}\right) \quad \text{since } \frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin\theta$$

Step 2 : To express Fourier series.

$$x(t) = \sum_{k=-\infty}^{\infty} X(k)e^{jk\omega_0 t} \text{ By definition of Fourier series}$$

Putting for $X(k)$, $x(t) = \sum_{k=-\infty}^{\infty}\frac{A}{k\pi}\sin\left(\frac{k\omega_0\tau}{2}\right)e^{jk\omega_0 t}$

Q.14 Find out the exponential Fourier series for impulse train shown in Fig. Q.14.1. Also plot its magnitude and phase spectrum.

[SPPU : May-06, 09, Dec.-17, June-22 Marks 6]

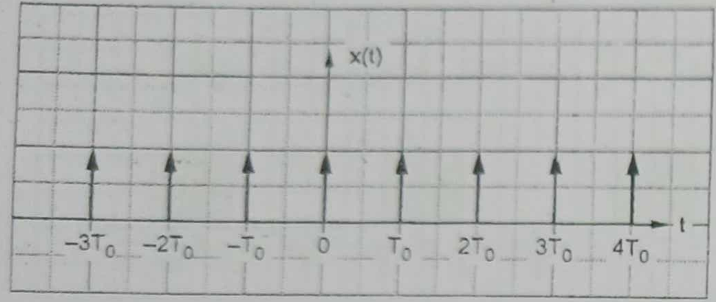


Fig. Q.14.1 Impulse train

Ans. : Fourier coefficients are given as,

$$X(k) = \frac{1}{T}\int_{<T>} x(t)e^{-jk\omega_0 t}\,dt$$

In Fig. Q.14.1 observe that the period of $x(t)$ is T_0 and $x(t) = \delta(t)$. i.e.,

$$X(k) = \frac{1}{T_0}\int_{<T>} \delta(t)e^{-jk\omega_0 t}\,dt$$

Here use shifting property of impulse function, $\int_{-\infty}^{\infty} x(t)\delta(t-t_0) = x(t_0)$

with $x(t) = e^{-jk\omega_0 t}$ and $t_0 = 0$ with $T_0 \to \infty$. i.e.,

$$X(k) = \frac{1}{T_0}e^{-jk\omega_0 t_0} = \frac{1}{T_0}, \text{ with } t_0 = 0,\ e^{-jk\omega_0 0} = e^0 = 1$$

Hence Fourier series is given as,

$$x(t) = \sum_{k=-\infty}^{\infty} X(k)e^{jk\omega_0 t} = \frac{1}{T_0}\sum_{k=-\infty}^{\infty} e^{jk\omega_0 t}$$

Q.15 Obtain complex exponential Fourier coefficients for the signal $x(t) = 4+2\cos 3t+3\sin 4t$. Also calculate the total power of x(t).

Ans. : Compare given equation with $x(t) = A+B\cos 2\pi f_1 t+C\sin 2\pi f_2 t$

$\therefore \quad 2\pi f_1 t = 3t \Rightarrow f_1 = \frac{3}{2\pi} = \frac{1}{T_1}$ or $T_1 = \frac{2\pi}{3}$

and $\quad 2\pi f_2 t = 4t \Rightarrow f_2 = \frac{4}{2\pi} = \frac{1}{T_2}$ or $T_2 = \frac{2\pi}{4}$

a) To obtain period T and ω_0

Step 1 : Multiply T_1 and T_2 by a number such as 12 so that their denominator will be '1'.

i.e. $\quad T_1 = \frac{2\pi}{3}\times 12 = 8\pi \quad$ and $\quad T_2 = \frac{2\pi}{4}\times 12 = 6\pi$

Step 2 : Now find out least common multiple of new periods T_1 and T_2. Here least common multiple of 8π and 6π is 24π.

Step 3 : Now divide the least common multiple by a number 12 (by which T_1 and T_2 were multiplied in step 1) i.e.,

$$T = \frac{24\pi}{12} = 2\pi \text{ is the period of } x(t)$$

We have,

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

b) To expand $x(t)$

We know that $\quad \cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$ and $\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}$,

then $x(t)$ can be written as,

$$x(t) = 4+2\cdot\frac{e^{j3t}+e^{-j3t}}{2}+3\cdot\frac{e^{j4t}-e^{-j4t}}{2j}$$

$$= 4+e^{j3t}+e^{-j3t}+\frac{3}{2j}e^{j4t}-\frac{3}{2j}e^{-j4t}$$

$$= 4+e^{j3\omega_0 t}+e^{-j3\omega_0 t}+\frac{3}{2j}e^{j4\omega_0 t}-\frac{3}{2j}e^{-j4\omega_0 t}$$

since $\omega_0 = 1$ ... (Q.15.1)

c) To obtain Fourier coefficients

We have, $\quad x(t) = \sum_{k=-\infty}^{\infty} X(k)e^{jk\omega_0 t}$

for $k = 4$, $\quad = \sum_{k=-4}^{4} X(k)e^{jk\omega_0 t}$

$$= X(-4)e^{-j4\omega_0 t} + X(-3)e^{-j3\omega_0 t} + X(-2)e^{-j2\omega_0 t} + X(-1)e^{-j\omega_0 t}$$
$$+ X(0) + X(1)e^{j\omega_0 t} + X(2)e^{j2\omega_0 t} + X(3)e^{j3\omega_0 t} + X(4)e^{j4\omega_0 t}$$

Comparing above equation with equation (Q.15.1) we get,

$$X(0) = 4$$
$$X(-1) = X(1) = X(-2) = X(2) = 0$$
$$X(-3) = X(3) = 1$$
$$X(-4) = -\frac{3}{2j} = j\frac{3}{2} \quad X(4) = \frac{3}{2j} = -j\frac{3}{2}$$

d) To obtain total power

By Parseval's theorem,

$$P = \sum_{k=-\infty}^{\infty} |X(k)|^2$$
$$= |X(-4)|^2 + |X(-3)|^2 + |X(-2)|^2 + |X(-1)|^2 + |X(0)|^2$$
$$+ |X(1)|^2 + |X(2)|^2 + |X(3)|^2 + |X(4)|^2$$

Putting values, $= \left(\frac{3}{2}\right)^2 + (1)^2 + 0 + 0 + (4)^2 + 0 + 0 + (1)^2 + \left(-\frac{3}{2}\right)^2$

$$= \frac{9}{4} + 1 + 16 + 1 + \frac{9}{4} = 22.5 \text{ W}$$

Q.16 Draw the magnitude and phase spectrum of the signal :
$x(t) = 5\cos(2\pi 10t + 30) - 10\cos(2\pi 20t + 60)$

[SPPU : Dec.-16, Marks 6]

Ans. : We know that,

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

Hence given equation can be written as,

$$x(t) = 5 \cdot \frac{e^{j(2\pi 10t + 30)} + e^{-j(2\pi 10t + 30)}}{2} - 10 \cdot \frac{e^{j(2\pi 20t + 60)} + e^{-j(2\pi 20t + 60)}}{2}$$
$$= 2.5e^{j2\pi 10t} \cdot e^{j30^\circ} + 2.5e^{j2\pi 10t} \cdot e^{-j30^\circ}$$
$$- 5e^{j2\pi 20t} \cdot e^{j60^\circ} - 5e^{-j2\pi 20t} \cdot e^{-j60^\circ}$$

Here note that one of the frequency of cosine wave is 10 Hz and other frequency is 20 Hz. Hence fundamental frequency is $f_0 = 10$ Hz.

$$\therefore \quad x(t) = A_1 e^{j2\pi f_0 t} \cdot e^{j\phi_1} + A_1 e^{-j2\pi f_0 t} \cdot e^{-j\phi_1}$$
$$+ A_2 e^{j2\pi f_0 t} \cdot e^{j\phi_2} + A_2 e^{-j2\pi f_0 t} \cdot e^{-j\phi_2}$$

Here $A_1 = 2.5$ V,

$f_0 = 10$ Hz,

$\phi_1 = 30^\circ$

and $A_2 = -5$ V,

$\phi_2 = 60^\circ$.

Fig. Q.16.1 shows the magnitude and phase spectrum based on above values.

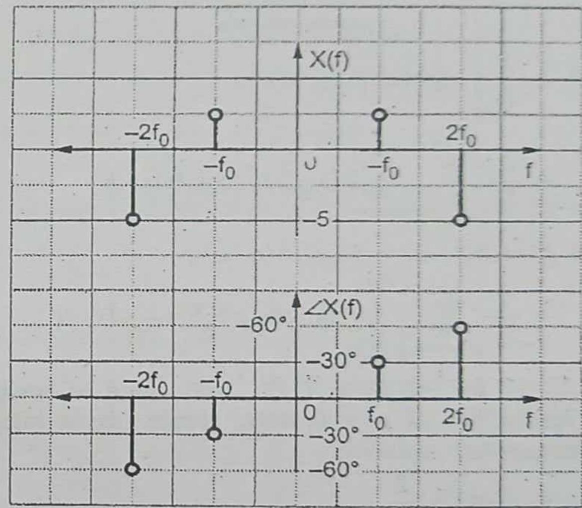


Fig. Q.16.1 : Magnitude and phase spectrum

Q.17 Find and sketch the trigonometric fourier series of train of impulse defined as :

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$

[SPPU : May-18, Marks 6]

Ans. :

$$a_0 = \frac{1}{T}\int_{<T>} x(t)\,dt$$

Here $T = T_s$ i.e. period of impulse train. Hence,

$$a_0 = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \sum_{k=-\infty}^{\infty}\delta(t-kT_s)\,dt$$

$$a_0 = \frac{1}{T_s}\sum_{k=-\infty}^{\infty}\underbrace{\int_{-T_s/2}^{T_s/2} d(t-kT_s)\,dt}_{\text{Value of this integration is '1', by property of impulse function}} = \frac{1}{T_s}\times 1 = \frac{1}{T_s}$$

$$a(k) = \frac{2}{T}\int_{<T>} x(t)\cos k\omega_0 t\,dt$$

$$= \frac{2}{T_s}\int_{-T_s/2}^{T_s/2}\sum_{p=-\infty}^{\infty}\delta(t-pT_s)\cos k\omega_0 t\,dt$$

Here let us use $f(t)\cdot\delta(t-a) = f(a)\cdot\delta(t-a)$,

$$= \frac{2}{T_s}\sum_{p=-\infty}^{\infty}\int_{-T_s/2}^{T_s/2} \sin k\omega_0 pT_s\delta(t-pTs)\,dt$$

Here $\delta(t-pT_s) = 1$ for $-T_s/2$ to $T_s/2$ And value of $\cos k\omega_0 pT_s = 1$, since above impulse lies at p = 0 only. Hence above integral has the value of 1. Hence,

$$a(k) = \frac{2}{T_s}\times 1 = \frac{2}{T_s}$$

$$b(k) = \frac{2}{T}\int_{<T>} x(t)\sin k\omega_0 t\,dt$$

$$= \frac{2}{T_s}\int_{-T_s/2}^{T_s/2}\sum_{p=-\infty}^{\infty}\delta(t-pT_s)\sin k\omega_0 t\,dt$$

$$= \frac{2}{T_s}\sum_{p=-\infty}^{\infty}\int_{-T_s/2}^{T_s/2} \sin k\omega_0 pT_s\delta(t-pTs)\,dt$$

Here $\delta(t-pT_s) = 1$ for $-T_s/2$ to $T_s/2$. And value of $\sin k\omega_0 pT_s = 0$, since above impulse lies at p = 0 only. Hence above integral has the value of 0. Hence,

$$b(k) = \frac{2}{T_s}\times 0 = 0$$

Then trigonometric fourier series is given as,

$$x(t) = a(0) + \sum_{k=1}^{\infty} a(k)\cos k\omega_0 t + \sum_{k=1}^{\infty} b(k)\sin k\omega_0 t$$

$$= \frac{1}{T_s} + \sum_{k=1}^{\infty}\frac{2}{T_s}\cos k\omega_0 t + 0$$

$$= \frac{1}{T_s}\left\{1 + \sum_{k=1}^{\infty} 2\cos k\omega_0 t\right\}$$

3.5 : DT Fourier Series and its Properties

Important Points to Remember

- The Discrete Time Fourier Series (DTFS) is used for DT signals. It is defined as,

$$x(n) = \sum_{k=<N>} X(k)e^{jk\Omega_0 n} \quad \text{(Synthesis equation)}$$

where, $$X(k) = \frac{1}{N}\sum_{n=<N>} x(n)\,e^{-jk\Omega_0 n} \quad \text{(Analysis equation)} \qquad ...(3.1)$$

- Here $n = <N>$ indicates summation over one period of 'N' samples.
- DTFS equation is a finite series over 'N' sampled. Hence DTFS is always convergent.
- DTFs is periodic and linear.

- If x (n) $\xleftrightarrow{DTFs}$ $X(k)$, then time and frequency shift properties for DTFs are expressed as follows :

$$x(n-n_0) \xleftrightarrow{DTFS} e^{-jk\Omega_0 n_0} X(k) \text{ and}$$

$$e^{jk_0\Omega_0 n}\, x(n) \xleftrightarrow{DTFS} X(k-k_0)$$

Q.18 State and prove convolution in time domain property of DTFS.

[SPPU : June-22, Dec.-22, Marks 2]

Ans. : If $x(n) \xleftrightarrow{DTFS} X(k)$ and $y(n) \xleftrightarrow{DTFS} Y(k)$, then

$$z(n) = x(n) * y(n) \xleftrightarrow{DTFS} Z(k) = N\, X(k)\, Y(k) \quad ...(Q.18.1)$$

Proof : $Z(k) = \frac{1}{N} \sum_{n=<N>} z(n) e^{-jk\Omega_0 n}$ by definition

$$= \frac{1}{N} \sum_{n=<N>} x(n) * y(n) e^{-jk\Omega_0 n}$$

Here $x(n) * y(n) = \sum_{l=<N>} x(l)\, y(n-l)$, then we have,

$$Z(k) = \frac{1}{N} \sum_{n=<N>} \sum_{l=<N>} x(l)\, y(n-l) e^{-jk\Omega_0 n}$$

Changing the order of summations,

$$Z(k) = \frac{1}{N} \sum_{l=<N>} x(l) \sum_{n=<N>} y(n-l) e^{-jk\Omega_0 n}$$

Putting $n - l = m$ for second summation will just shift the limits by 'l' but summation will be over a period of 'N' samples. Hence limits need not be changed. i.e.,

$$Z(k) = \frac{1}{N} \sum_{l=<N>} x(l) \sum_{m=<N>} y(m) e^{-jk\Omega_0 (m+l)}$$

$$= \frac{1}{N} \sum_{l=<N>} x(l) \sum_{m=<N>} y(m) e^{-jk\Omega_0 m} \cdot e^{-jk\Omega_0 l}$$

$$= \frac{1}{N} \left[\sum_{l=<N>} x(l) e^{-jk\Omega_0 l} \sum_{m=<N>} y(m) e^{-jk\Omega_0 m} \right]$$

Rearrange the above equation as,

$$Z(k) = N \left[\frac{1}{N} \sum_{l=<N>} x(l) e^{-jk\Omega_0 l} \right] \left[\frac{1}{N} \sum_{m=<N>} y(m) e^{-jk\Omega_0 m} \right]$$

$$= N\, X(k) \cdot Y(k)$$

Significance : Convolution in time domain becomes multiplication of Fourier coefficients.

Q.19 State and prove Parseval's theorem for DTFS.

Ans. : If $x(n) \xleftrightarrow{DTFS} X(k)$, then

$$P = \sum_{k=<N>} |X(k)|^2 \quad ...(Q.19.1)$$

Here, 'P' is the average power of periodic sequence $x(n)$.

Proof : Average power of the sequence is given as,

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

For periodic sequence with period 'N' above equation can be written as,

$$P = \frac{1}{N} \sum_{n=<N>} |x(n)|^2$$

$$= \frac{1}{N} \sum_{n=<N>} x(n) x^*(n) \text{, since } x(n)x^*(n) = |x(n)|^2 \quad ...(Q.19.2)$$

We know that,

$$x(n) = \sum_{k=<N>} X(k) e^{jk\Omega_0 n} \quad \text{Synthesis equation of DTFS}$$

$$\therefore \quad x^*(n) = \sum_{k=<N>} [X(k) e^{jk\Omega_0 n}]^*$$

$$= \sum_{k=<N>} X^*(k) e^{-jk\Omega_0 n}$$

Putting above equation of $x^*(n)$ in equation (Q.19.2) we get,

$$P = \frac{1}{N} \sum_{n=<N>} x(n) \sum_{k=<N>} X^*(k) e^{-jk\Omega_0 n}$$

Changing the order of summation we get,

$$P = \sum_{k=<N>} X^*(k)\left[\frac{1}{N}\sum_{n=<N>} x(n)e^{-jk\Omega_0 n}\right]$$

$$= \sum_{k=<N>} X^*(k)X(k) \text{ since the quantity insider the brackets is } X(k)$$

$$= \sum_{k=<N>} |X(k)|^2$$

Significance : The total average power of sequence $x(n)$ is obtained by summation of powers of all the Fourier coeffcients.

Q.20 Find the DTFS coefficients of the signal shown in Fig. Q.20.1. Also calculate total power using Parseval's theorem.

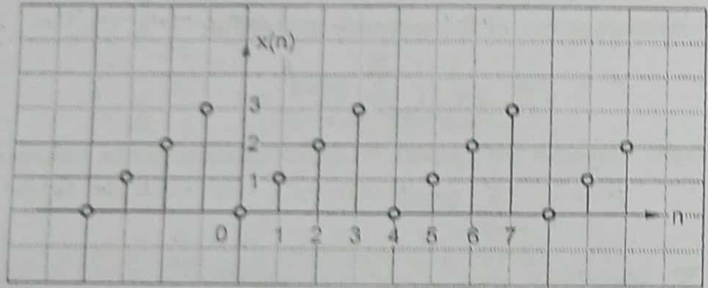


Fig. Q.20.1 Sequence

Ans. : Step 1 : The sequence repeats at $N = 4$.

Hence $\Omega_0 = \frac{2\pi}{N} = \frac{2\pi}{4} = \frac{\pi}{2}$.

Step 2 : Fourier coefficients are given as,

$$X(k) = \frac{1}{N}\sum_{n=<N>} x(n)e^{-jk\Omega_0 n}$$

$$= \frac{1}{4}\sum_{n=0}^{3} x(n)e^{-jk\frac{\pi}{2}n}$$

$$= \frac{1}{4}\left[x(0)+x(1)e^{-jk\frac{\pi}{2}}+x(2)e^{-jk\pi}+x(3)e^{-jk\frac{3\pi}{2}}\right]$$

Putting values of $x(n)$

$$X(k) = \frac{1}{4}\left[e^{-jk\frac{\pi}{2}}+2e^{-jk\pi}+3e^{-jk\frac{3\pi}{2}}\right]$$

$$= \frac{1}{4}\left[\left(e^{-j\frac{\pi}{2}}\right)^k+2\left(e^{-j\pi}\right)^k+3\left(e^{-j\frac{3\pi}{2}}\right)^k\right] \quad \text{... (Q.20.1)}$$

We have,

$$e^{-j\frac{\pi}{2}} = \cos\frac{\pi}{2}-j\sin\frac{\pi}{2} = -j$$

$$e^{-j\pi} = \cos\pi-j\sin\pi = -1$$

$$e^{-j\frac{3\pi}{2}} = \cos\frac{3\pi}{2}-j\sin\frac{3\pi}{2} = j$$

Putting values in equation (Q.20.1) we get following DTFS coefficients

$$X(k) = \frac{1}{4}[(-j)^k+2(-1)^k+3(j)k]$$

$$\therefore \quad X(0) = \frac{1}{4}[1+2+3] = \frac{6}{4} = \frac{3}{2}$$

$$X(1) = \frac{1}{4}[-j-2+3j] = \frac{-2+2j}{4} = \frac{-1}{2}+j\frac{1}{2}$$

$$X(2) = \frac{1}{4}[-1+2-3] = \frac{-2}{4} = -\frac{1}{2}$$

$$X(3) = \frac{1}{4}[-j^3+2(-1)^3+3j^3] = \frac{j-2-3j}{4} = -\frac{1}{2}-j\frac{1}{2}$$

Step 3 : To obtain total power

$$P = \sum_{k=<N>} |X(k)|^2 \quad \text{By equation (Q.19.1)}$$

$$= |X(0)|^2+|X(1)|^2+|X(2)|^2+|X(3)|^2 \quad \text{for } k = 0 \text{ to } 3$$

$$= \left(\frac{3}{2}\right)^2+\left[\left(-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2\right]+\left(-\frac{1}{2}\right)^2+\left[\left(-\frac{1}{2}\right)^2+\left(-\frac{1}{2}\right)^2\right]$$

$$= \frac{7}{2}$$

Q.21 For a discrete time signal x(n), find fourier series coefficients if, $x(n) = \sin\left(\frac{\pi}{4}n\right) + \cos\left(\frac{\pi}{8}n\right)$

[SPPU : Dec.-03, Marks 5]

Ans. : The given signal can be expressed as,

$$x(n) = \sin\left(2\pi\frac{1}{8}n\right) + \cos\left(2\pi\frac{1}{16}n\right)$$

Hence $f_1 = \frac{1}{8}$ and $f_2 = \frac{1}{16}$

$\therefore$ $N_1 = 8$ and $N_2 = 16$

The least common multiple of N_1 and N_2 is 16. Hence $x(n)$ has the period of $N = 16$. We know that $\sin\theta = \frac{1}{j2}e^{j\theta} - \frac{1}{j2}e^{-j\theta}$ and $\cos\theta = \frac{1}{2}e^{j\theta} + \frac{1}{2}e^{-j\theta}$. Therefore $x(n)$ can be written as,

$$x(n) = \frac{1}{j2}e^{j\pi n/4} - \frac{1}{j2}e^{-j\pi n/4} + \frac{1}{2}e^{j\pi n/8} + \frac{1}{2}e^{-j\frac{\pi n}{8}} \quad \text{... (Q.21.1)}$$

Fourier series is given as

$$x(n) = \sum_{k=<N>} X(k)e^{jk\Omega_0 n}$$

Here $\Omega_0 = \frac{2\pi}{N} = \frac{2\pi}{16} = \frac{\pi}{8}$

$$\therefore \quad x(n) = \sum_{k=<N>} X(k)e^{j\pi kn/8}$$

Let us evaluate above equation for $k = -2$ to 13.

$$x(n) = X(-2)e^{-j\pi n/4} + X(-1)e^{-j\pi n/8} + X(0)e^0 + X(1)e^{j\pi n/8} + X(2)e^{j\pi n/4} + X(3)e^{j3\pi n/8} + \ldots\ldots + X(13)e^{j13\pi n/8}$$

Comparing above equation with equation (Q.21.1),

$$X(-2) = -\frac{1}{j2},\ X(-1) = \frac{1}{2},\ X(0) = 0,\ X(1) = \frac{1}{2},\ X(2) = \frac{1}{j2}$$

And $X(3) = X(4) = X(5) = X(6) = \ldots\ldots\ldots = X(13) = 0$

And using the property $X(k + N) = X(k)$ we can find other Fourier coefficients. Above coefficients are for one period of $x(n)$. They repeat after $N = 16$ samples.

Q.22 State and explain the following properties of DTFs

i) Linearity [SPPU : June-22, Marks 2]

ii) Time shift [SPPU : Dec.-22, Marks 2].

Ans. : i) Linearity

If $x(n) \xleftrightarrow{DTFS} X(k)$ and $y(n) \xleftrightarrow{DTFS} Y(k)$.

then, $\boxed{z(n) = ax(n) + by(n) \xleftrightarrow{DTFS} Z(k) = aX(k) + bY(k)}$ ... (Q.22.1)

Proof :

$$Z(k) = \frac{1}{N}\sum_{n=<N>} z(n)e^{-jk\Omega_0 n}$$

$$= \frac{1}{N}\sum_{n=<N>} [a\,x(n) + b\,y(n)]\ e^{-jk\Omega_0 n}$$

$$= a\frac{1}{N}\sum_{n=<N>} x(n)e^{-jk\Omega_0 n} + b\frac{1}{N}\sum_{n=<N>} y(n)e^{-jk\Omega_0 n}$$

$$= a\,X(k) + b\,Y(k)$$

ii) Time Shift

If $x(n) \xleftrightarrow{DTFS} X(k)$ then,

$$\boxed{y(n) = x(n-n_0) \xleftrightarrow{DTFS} Y(k) = e^{-jk\Omega_0 n_0} X(k)} \quad \text{... (Q.22.2)}$$

Proof :

$$Y(k) = \frac{1}{N}\sum_{n=<N>} y(n)e^{-jk\Omega_0 n} = \frac{1}{N}\sum_{n=<N>} x(n-n_0)e^{-jk\Omega_0 n}$$

Let $n - n_0 = m$, then

$$Y(k) = \frac{1}{N}\sum_{m=<N>} x(m)e^{-jk\Omega_0 (m+n_0)}$$

$$= e^{-jk\Omega_0 n_0} \frac{1}{N} \sum_{m=<N>} x(m) e^{-jk\Omega_0 m}$$

$$= e^{-jk\Omega_0 n_0} X(k)$$

FORMULAE AT A GLANCE

i) For orthogonal signals, $\int_{t_1}^{t_2} f(t) \cdot x(t)\, dt = 0$

ii) Trigonometric fourier series :

$$x(t) = a(0) + \sum_{k=1}^{\infty} a(k) \cos k\omega_0 t + \sum_{k=1}^{\infty} b(k) \sin k\omega_0 t$$

where $a(0) = \frac{1}{T} \int_{<T>} x(t)\, dt$

$$a(k) = \frac{2}{T} \int_{<T>} x(t) \cos k\omega_0 t\, dt$$

$$b(k) = \frac{2}{T} \int_{<T>} x(t) \sin k\omega_0 t\, dt \quad \ldots(3.2)$$

iii) Compact trigomometric fourier series :

$$x(t) = D(0) + \sum_{k=1}^{\infty} D(k) \cos (k\omega_0 t + \phi(k))$$

where $D(0) = a_0 = \frac{1}{T} \int_{<T>} x(t)\, dt$ ...(3.3)

$$D(k) = \sqrt{a(k)^2 + b(k)^2} \text{ and } \phi(k) = -\tan^{-1}\left(\frac{b(k)}{a(k)}\right)$$

iv) Exponential fourier series :

$$x(t) = \sum_{k=-\infty}^{\infty} X(k)\, e^{jk\omega_0 t} \quad \text{(synthesis equation)}$$

where $X(k) = \frac{1}{T} \int_{<T>} e^{-jk\omega_0 t}\, dt$ (analysis equation) ...(3.4)

v) Discrete time fourier series :

$$x(n) = \sum_{k=<N>} X(k) e^{jk\Omega_0 n} \quad \text{(Synthesis equation)}$$

where, $X(k) = \frac{1}{N} \sum_{n=<N>} x(n)\, e^{-jk\Omega_0 n}$ (Analysis equation) ...(3.5)

END...

Unit IV

4 Fourier Transform

Fourier transform properties

Sr. No.	Property	Mathematical description
1.	Linearity	$ax_1(t) + bx_2(t) \leftrightarrow aX_1(\omega) + bX_2(\omega)$ a and b are constants.
2.	Time scaling	$x(at) \leftrightarrow \frac{1}{\lvert a \rvert} X\left(\frac{\omega}{a}\right)$ a is constant.
3.	Duality	$X(t) \leftrightarrow 2\pi x(-\omega)$
4.	Time shifting	$x(t - t_0) \leftrightarrow X(\omega)\, e^{-j\omega t_0}$
5.	Frequency shifting	$e^{j\beta}\, x(t) \leftrightarrow X(\omega - \beta)$
6.	Area under $x(t)$	$\int_{-\infty}^{\infty} x(t)\, dt = X(0)$ i.e. FT at $\omega = 0$
7.	Area under $X(\omega)$	$\int_{-\infty}^{\infty} X(\omega)\, d\omega = x(0)$ i.e. IFT at $t = 0$
8.	Differentiation in time domain	$\frac{d}{dt} x(t) \leftrightarrow j\omega\, X(\omega)$
9.	Integration in time domain	$\int_{-\infty}^{t} x(\tau)\, d\tau \leftrightarrow \frac{1}{j\omega} X(\omega)$
10.	Conjugate functions	If $x(t) \leftrightarrow X(\omega)$ then $x^*(t) \leftrightarrow X^*(-\omega)$

11.	Multiplication in time domain	$x_1(t)\, x_2(t) \leftrightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\lambda)\, X_2(\omega - \lambda)\, d\lambda$
12.	Convolution in time domain	$\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t - \tau)\, d\tau \leftrightarrow X_1(\omega)\, X_2(\omega)$

Table 4.1 Properties of Fourier transform

Fourier transform pairs

Sr. No.	Name of the signal	Mathematical representation	Fourier transform
1.	Rectangular pulse of width T	$rect\left(\frac{t}{2T}\right)$	$2T\, sinc\left(\frac{\omega T}{\pi}\right)$
2.	Sinc pulse with zero crossing at $\frac{n}{2W}$	$sinc(2Wt)$	$\frac{1}{2W}\, rect\left(\frac{f}{2W}\right)$
3.	Decaying exponential signal for $t > 0$	$e^{-at}\, u(t)\ a > 0$	$\frac{1}{a + j\omega}$
4.	Double exponential signal	$e^{-a\lvert t \rvert}, a > 0$	$\frac{2a}{a^2 + (\omega)^2}$
5.	Triangular	$\begin{cases} 1 - \frac{\lvert t \rvert}{T}, & \lvert t \rvert < T \\ 0, & \lvert t \rvert \geq T \end{cases}$	$T\, sinc^2\left(\frac{\omega T}{\pi}\right)$
6.	Unit impulse	$\delta(t)$	1
7.	DC signal	1	$2\pi\delta(\omega)$
8.	Delayed unit impulse	$\delta(t - t_0)$	$e^{-j\omega t_0}$
9.	Phasor of frequency ω_c	$e^{j\omega_c t}$	$\delta(\omega - \omega_c)$

10.	Cosine wave signal	$\cos(\omega_0 t)$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
11.	Sine wave signal	$\sin(\omega_0 t)$	$\frac{\pi}{j}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
12.	Function	$sgn(t)$	$\frac{2}{j\omega}$
13.	Unit step	$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
14.	Impulse train or sampling function	$\sum_{m=-\infty}^{\infty}\delta(t-mT_0)$	$\omega_0\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_0)$

Table 4.2 Summary of Fourier transform pairs

4.1 : Forier Transform Representation of a Periodic CT Signals

Important Points to Remember

- Fourier transform is defined for continuous time signals. It is given as,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \text{ or } X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt$$

- Inverse Fourier transform is given as,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}\, df$$

Q.1 Define Fourier transform pair.

Ans. : Definition of Fourier transform :

The Fourier transform of $x(t)$ is defined as,

Fourier transform :

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \quad \text{or} \quad X(t) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt \quad \text{...(Q.1.1)}$$

Here '$x(t)$' is time domain representation of the signal and '$X(\omega)$' or '$X(f)$' is frequency domain representation of the signal, 'ω' is the frequency. Sometimes $X(\omega)$ is also written as $X(j\omega)$.

Similarly $x(t)$ can be obtained from $X(\omega)$ by inverse Fourier transform i.e.,

Inverse Fourier transform :

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft}\, dt \quad \text{...(Q.1.2)}$$

A Fourier transform pair is represented as,

$$x(t) \xleftrightarrow{FT} X(\omega) \quad \text{or} \quad x(t) \xleftrightarrow{FT} X(f)$$

Q.2 State the Dirichlet conditions to be satisfied for existance of Fourier transform. ☞ [SPPU : Dec.-19, Marks 3, June-22, Marks 5]

Ans. : i) Single valued property : $x(t)$ must have only value at any time instant over a finite time interval T.

ii) Finite discontinuities : $x(t)$ should have at the most finite number of discontinuities over a finite time interval T.

iii) Finite peaks : The signal $x(t)$ should have finite number of maxima and minima over a finite time interval T.

iv) Absolute integrability : $x(t)$ should be absolutely integrable i.e.,

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

4.2 : Fourier Transform of Standard CT Signals

Important Points to Remember

- Standard CT signals such as unit impulse, unit step, signum, exponential and sinusoidal signals have fourier transforms.
- These fourier transforms are used as standard relations in signal analysis problems.

Q.3 Obtain the Fourier transform of following signals and plot their magnitude / phase spectrum.

i) $x(t) = e^{-at}u(t)$ ii) $x(t) = e^{at}u(-t)$

iii) $x(t) = e^{-a|t|}$ iv) $x(t) = e^{-a|t|}\,\text{sgn}(t)$

[SPPU : Dec.-04, Marks 4; May-07, Marks 6, Dec.-17,19, Marks 3, June-22, Marks 6]

Ans. : i) $x(t) = e^{-at}u(t)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{-at}\cdot e^{-j\omega t}\,dt$$

Since $u(t) = 1$ for $t \geq 0$

$$= \int_{0}^{\infty} e^{-(a+j\omega)t}\,dt = \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_0^{\infty} = \frac{1}{a + j\omega}$$

Thus,

$$\boxed{e^{-at}u(t) \overset{FT}{\longleftrightarrow} \frac{1}{a+j\omega}} \qquad \ldots (Q.3.1)$$

Magnitude and phase spectrum

$$X(\omega) = \frac{1}{a+j\omega}\times\frac{a-j\omega}{a-j\omega} \quad \text{By rearranging}$$

$$= \frac{a}{a^2+\omega^2} - j\frac{\omega}{a^2+\omega^2}$$

$$\therefore |X(\omega)| = \sqrt{\frac{a^2+\omega^2}{(a^2+\omega^2)^2}} = \sqrt{\frac{1}{a^2+\omega^2}}$$

$$\angle X(\omega) = \tan^{-1}\frac{\left(-\frac{\omega}{a^2+\omega^2}\right)}{\left(\frac{a}{a^2+\omega^2}\right)} = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

Fig. Q.3.1 shows the magnitude/phase spectrum for

a = 1 and varying 'ω'.

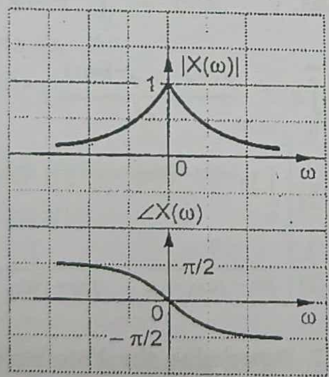


Fig. Q.3.1 Magnitude / phase spectrum

ii) $x(t) = e^{at}u(-t)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} e^{at}u(-t)e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{0} e^{at}e^{-j\omega t}\,dt, \quad \text{Since } u(-t) = 1 \text{ for } 0 \text{ to} -\infty$$

$$= \int_{-\infty}^{0} e^{(a-j\omega)t}\,dt = \frac{1}{a-j\omega}$$

Thus,

$$\boxed{e^{at}u(-t) \overset{FT}{\longleftrightarrow} \frac{1}{a-j\omega}} \qquad \ldots (Q.3.2)$$

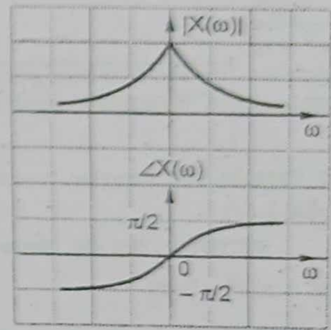


Fig. Q.3.2 Magnitude/phase spectrum

iii) $x(t) = e^{-a|t|}$

Here,

$$x(t) = \begin{cases} e^{-at} & \text{for } t > 0 \\ 1 & \text{for } t = 0 \\ e^{at} & \text{for } t < 0 \end{cases}$$

$$\therefore \quad X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,d\omega$$

$$= \int_{-\infty}^{0-} e^{at}e^{-j\omega t}\,dt + \int_{0-}^{0+} 1\cdot e^{-j\omega t}\,dt + \int_{0+}^{\infty} e^{-at}e^{-j\omega t}\,dt$$

$$= \left[\frac{e^{(a-j\omega)t}}{a-j\omega}\right]_{-\infty}^{0} + \left[e^{0+} - e^{0-}\right] + \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_0^{\infty} = \frac{2a}{a^2+\omega^2}$$

Thus,

$$\boxed{e^{-a|t|} \overset{FT}{\longleftrightarrow} \frac{2a}{a^2+\omega^2}} \qquad \ldots (Q\ 3.3)$$

Fig. Q.3.3 shows the magnitude and phase response.

iv) $x(t) = e^{-a|t|}\,\text{sgn}(t)$

Here,

$$\text{sgn}(t) = \begin{cases} 1 & \text{for } t > 0 \\ -1 & \text{for } t < 0 \end{cases}$$

$$\therefore \quad x(t) = \begin{cases} e^{-at} & \text{for } t>0 \\ -e^{at} & \text{for } t<0 \end{cases}$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}$$

$$= \int_{-\infty}^{0-} -e^{at} e^{-j\omega t} dt + \int_{0+}^{\infty} e^{-at} \cdot e^{-j\omega t} dt$$

$$= \frac{-j2\omega}{a^2+\omega^2}$$

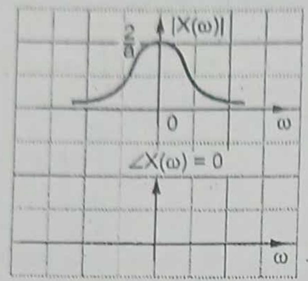


Fig. Q.3.3 Magnitude/phase spectrum

Thus, $$e^{-a|t|} \operatorname{sgn}(t) \xleftrightarrow{FT} \frac{-j2\omega}{a^2+\omega^2} \quad \text{... (Q.3.4)}$$

Q.4 Obtain the Fourier transforms of following functions :
$x(t) = \delta(t)$

[SPPU : Dec.-07, Marks 2; May-11, June-22, Marks 5, Dec.-17, Marks 3]

Ans. : $x(t) = \delta(t)$

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt$$

Shifting property of impulse function states $\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0)$.

Here with $t_0 = 0$ in above equation,

$$X(\omega) = e^{-j\omega t}\Big|_{t=0} = 1$$

Thus $$\delta(t) \xleftrightarrow{FT} 1 \quad \text{... (Q.4.1)}$$

Q.5 Obtain the Fourier transform of following signals.

i) x(t) = $\sin \omega_c t \ u(t)$ **[SPPU : Dec.-16, Marks 6]**

ii) $x(t) = \cos \omega_c t \ u(t)$

[SPPU : Dec.-04, Marks 6; Dec.-05,18, Marks 8, May-19, Marks 2, Dec.-19, Marks 3]

Ans. : i) $x(t) = \cos \omega_c t \ u(t)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{0}^{\infty} \cos \omega_c t \, e^{-j\omega t} dt, \quad \text{since } u(t) = 1 \text{ for } t \ge 0$$

$$= \int_{0}^{\infty} \frac{e^{j\omega_c t} + e^{-j\omega_c t}}{2} \cdot e^{-j\omega t} dt$$

$$= \frac{1}{2}\int_{0}^{\infty} \left[e^{-j(\omega-\omega_c)t} + e^{-j(\omega+\omega_c)t}\right] dt = \frac{2\omega}{j\left(\omega^2-\omega_c^2\right)} \quad \text{... (Q.5.1)}$$

ii) $x(t) = \sin \omega_c t \ u(t)$

$$X(\omega) = \int_{0}^{\infty} \sin \omega_c t \, e^{-j\omega t} dt = \int_{0}^{\infty} \frac{e^{j\omega_c t} - e^{-j\omega_c t}}{2j} \cdot e^{-j\omega t} dt$$

$$= \frac{1}{2j}\int_{0}^{\infty} \left[e^{-j(\omega-\omega_c)t} - e^{-j(\omega+\omega_c)t}\right] dt = -\frac{\omega_c}{\omega^2-\omega_c^2} \quad \text{... (Q.5.2)}$$

Q.6 Find Fourier transform of x(t) = e^{-at} sin $\omega_0 t \, u(t)$.

[SPPU : May-11, Marks 8, Dec.-14, Marks 2]

Ans. : $$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} e^{-at} \sin \omega_0 t \, u(t) \, e^{-j\omega t} \, dt$$

$$= \int_{0}^{\infty} e^{-at} \cdot \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} e^{-j\omega t} dt$$

$$= \frac{1}{2j}\left[\int_{0}^{\infty} [e^{-(j\omega - j\omega_0 + a)t} - e^{-(j\omega + j\omega_0 + a)t}]dt\right]$$

$$= \frac{1}{2j}\left[\frac{1}{j\omega - j\omega_0 + a} - \frac{1}{j\omega + j\omega_0 + a}\right] = \frac{\omega_0}{(j\omega+a)^2 + \omega_0^2}$$

Q.7 Determine the Fourier transform of the following signals.

x(t) = $e^{-3|t|}$ **[SPPU : Dec.-15, Marks 3]**

Ans. : Consider the standard FT relation,

$$e^{-a|t|} \xleftrightarrow{FT} \frac{2a}{a^2+\omega^2}$$

$$e^{-3|t|} \xleftrightarrow{FT} \frac{2\times 3}{3^2+\omega^2} = \frac{6}{9+\omega^2}$$

4.3 : Properties and their Significance

Important Points to Remember

- Time shift and frequency shift properties of CTPT are given as,

$x(t-t_0) \xleftrightarrow{FT} e^{-j\omega t_0} X(\omega)$ and $e^{j\beta t} x(t) \xleftrightarrow{FT} X(\omega-\beta)$

- Time scaling property is given as,

$x(at) \xleftrightarrow{FT} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

- The duality properly of Fourier transform states,

$X(t) \xleftrightarrow{FT} 2\pi x(-\omega)$

- Fourier transform of real and even signal is real. Similarly, Fourier transform of odd signal is imaginary.
- Parseval's theorem or Rayleigh's theorem for Fourier transform is given as,

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |X(f)|^2 df.$$

- Fourier transform of periodic signals is given as,

$$X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} X(k)\delta(\omega-k\omega_0)$$

Here '$X(k)$' are Fourier series coefficients of the given signal.

Q.8 State and prove following properties of CTFT.

i) Frequency differentation

ii) Time differentation

iii) Convolution [SPPU : June-22, Dec.-22, Marks 4, Dec.-18, Marks 3]

Ans. : i) Frequency differentiation

Statement : Differentiating the frequency spectrum is equivalent to multiplying the time domain signal by complex number $-jt$.

$$-jt\ x(t) \xleftrightarrow{FT} \frac{d}{d\omega}X(\omega) \quad \text{... (Q.8.1)}$$

Proof : $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$

$$\therefore \quad \frac{d}{d\omega}X(\omega) = \int_{-\infty}^{\infty} x(t)\frac{d}{d\omega}\left[e^{-j\omega t}\right]dt = \int_{-\infty}^{\infty} x(t)(-jt)\, e^{-j\omega t} dt$$

$$= -jt\int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt = -jt\ X(\omega)$$

ii) Time differentiation

Statement : Differentiation in time domain corresponds to multiplying by $j\omega$ in frequency domain. It accentuates high frequency components of the signal.

$$\frac{d\,x(t)}{dt} \xleftrightarrow{FT} j\omega X(\omega) \quad \text{... (Q.8.2)}$$

Proof : $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t} d\omega$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\left[\frac{d}{dt}e^{j\omega t}\right]d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, j\omega\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[j\omega\, X(\omega)\right] e^{j\omega t}\, d\omega$$

Thus Fourier transform is multiplied by '$j\omega$'.

iii) Convolution

Statement : A convolution operation is transformed to modulation in frequency domain.

$$z(t) = x(t) * y(t) \xleftrightarrow{FT} Z(\omega) = X(\omega)\cdot Y(\omega) \quad \text{... (Q.8.3)}$$

Proof : $Z(\omega) = \int_{-\infty}^{\infty} z(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty}[x(t) * y(t)]\, e^{-j\omega t}dt$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau\right] e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} y(t-\tau)\, e^{-j\omega t}\, dt\right] d\tau$$

Put $t-\tau = \alpha$, then $t = \tau+\alpha$.

$dt = d\alpha$, limits of integration will remain same.

$$\therefore \quad Z(\omega) = \int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} y(\alpha)e^{-j\omega(\tau+\alpha)}d\alpha\right] d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} y(\alpha)e^{-j\omega\tau} \cdot e^{-j\omega\alpha}\, d\alpha\right] d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}\, d\tau \int_{-\infty}^{\infty} y(\alpha)e^{-j\omega\alpha}\, d\alpha = X(\omega)\cdot Y(\omega)$$

Q.9 State and prove the following properties of CTFT (i) Time scaling (ii) Time shifting. ☞ [SPPU : Dec.-18,22, Marks 2, June-22, Marks 4]

Ans. : (i) Time scaling : Statement : Compression of a signal in time domain is equivalent to expansion in frequency domain and vice-versa. This time as well as frequency.

$$\boxed{y(t) = x(at) \xleftrightarrow{FT} Y(\omega) = \frac{1}{|a|}X\left(\frac{\omega}{a}\right)} \quad \text{...(Q.9.1)}$$

Proof : $Y(\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt$

Put at $= \tau$, then $t = \frac{\tau}{a}$

$\therefore \quad dt = \frac{1}{a}d\tau$ and limits will remain same.

$$\therefore \quad Y(\omega) = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\frac{\tau}{a}}\cdot\frac{1}{a}d\tau = \frac{1}{a}\int_{-\infty}^{\infty} x(\tau)e^{-j\left(\frac{\omega}{a}\right)^{\tau}} d\tau$$

$$= \frac{1}{a}X\left(\frac{\omega}{a}\right)$$

(ii) Time shifting : Statement : A shift of 't_0' in time domain is equivalent to introducing a phase shift of $(-\omega t_0)$. But amplitude remains same.

$$\boxed{y(t) = x(t-t_0) \xleftrightarrow{FT} Y(\omega) = e^{-j\omega t_0}X(\omega)} \quad \text{...(Q.9.2)}$$

Proof : $Y(\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} y(t-t_0)\, e^{-j\omega t}\, dt$

Put $t-t_0 = \tau$ then $t = \tau+t_0$.

$\therefore \quad dt = d\tau$ and integration limits will remain same.

$$Y(\omega) = \int_{-\infty}^{\infty} y(\tau)e^{-j\omega(\tau+t_0)}d\tau = \int_{-\infty}^{\infty} y(\tau)e^{-j\omega\tau}\cdot e^{-j\omega t_0}d\tau$$

$$= e^{-j\omega t_0}\int_{-\infty}^{\infty} y(\tau)e^{-j\omega\tau}d\tau = e^{-j\omega t_0}Y(\omega)$$

Q.10 State and prove Parseval's theorem for fourier transform.

☞ [SPPU : Dec.-18,22, June-22, Marks 2]

Ans. : Parseval's theorem or Rayleigh's theorem :

Statement : Energy of the signal can be obtained by interchanging its energy spectrum.

$$E = \int_{-\infty}^{\infty}|x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2 d\omega$$

$$= \int_{-\infty}^{\infty}|X(f)|^2 df \quad \text{...(Q.10.1)}$$

Proof : $E = \int_{-\infty}^{\infty}|x(t)|^2 dt = \int_{-\infty}^{\infty} x(t)x^* t\, dt$ ...(Q.10.2)

Inverse Fourier transform states that,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Taking conjugate of both the sides,

$$x^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x^*(\omega) e^{-j\omega t} d\omega$$

Putting above expression for $x^*(t)$ in equation (Q.10.2).

$$E = \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega) e^{-j\omega t}\right] d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega) \int_{-\infty}^{\infty} x(t) e^{-j\omega t} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)\cdot X(\omega)\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Since $\omega = 2\pi f$, $d\omega = 2\pi df$. Hence above equation becomes,

$$\therefore \quad E = \int_{-\infty}^{\infty} |X(f)|^2 df$$

Q.11 Obtain the Fourier transforms of following functions :

i) $x(t) = 1$ ☞[SPPU : Dec.-22, Marks 6]

ii) $x(t) = \text{sgn}(t)$ ☞[SPPU : May-11, Marks 5, Dec.-13,18, May-18, Marks 3, Dec.-14, Marks 2, Dec.-22, Marks 7]

iii) $x(t) = u(t)$

☞[SPPU : May-09, Marks 7; Dec.-10, Marks 6, Dec.-14, Marks 2, May-18, Makrs 3]

Ans. : i) $x(t) = 1$

We have, $\delta(t) \xleftrightarrow{FT} 1$

Duality property states $X(t) \xleftrightarrow{FT} 2\pi x(-\omega)$. Here let $X(t) = 1$, then $x(-\omega) = \delta(-\omega)$. Then we can write,

$$1 \xleftrightarrow{FT} 2\pi\,\delta(-\omega)$$

$$\boxed{1 \xleftrightarrow{FT} 2\pi\,\delta(\omega)} \quad \text{since } \delta(\omega) \text{ is even function} \qquad \text{... (Q.11.1)}$$

ii) $x(t) = \text{sgn}(t)$

The function sgn(t) can be written as,

$$x(t) = \text{sgn}(t) = 2u(t) - 1$$

Differentiating both sides,

$$\frac{dx(t)}{dt} = 2\frac{d}{dt}u(t)$$

$$\therefore \quad \frac{dx(t)}{dt} = 2\delta(t)$$

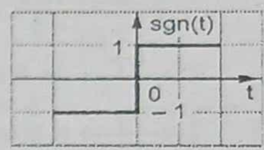


Fig. Q.11.1 sgn(*t*)

Taking Fourier transform of both sides,

$$j\omega X(\omega) = 2, \quad \text{since } \delta(t) \xleftrightarrow{FT} 1$$

or $$X(\omega) = \frac{2}{j\omega}$$

Thus $$\boxed{\text{sgn}(t) \xleftrightarrow{FT} \frac{2}{j\omega}} \qquad \text{... (Q.11.2)}$$

iii) $x(t) = u(t)$

We know that, $\text{sgn}(t) = 2u(t) - 1$

$$\therefore \quad u(t) = \frac{1}{2}\{1 + \text{sgn}(t)\}$$

Taking Fourier transform of both sides,

$$FT\{u(t)\} = \frac{1}{2}\left[FT\{1\} + FT\{\text{sgn}(t)\}\right]$$

$$\therefore u(t) \xleftrightarrow{FT} \frac{1}{2}\left[2\pi\,\delta(\omega) + \frac{2}{j\omega}\right]$$

... From equation (Q.11.1) and equation (Q.11.2)

$$\therefore \boxed{u(t) \xleftrightarrow{FT} \pi\delta(\omega) + \frac{1}{j\omega}} \qquad \text{(Q.11.3)}$$

Q.12 Obtain the Fourier transform of following signals.

i) $x(t) = \cos \omega_0 t$ ☞[SPPU : Dec.-19, Marks 3]

ii) $x(t) = \sin \omega_0 t$ ☞[SPPU : May-12, Marks 5]

Ans. : i) $x(t) = \cos \omega_0 t$

$$= \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} = \frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t} \quad \text{... (Q.12.1)}$$

We have $1 \overset{FT}{\longleftrightarrow} 2\pi\, \delta(\omega)$ ... By equation (Q.11.1)

$\therefore \quad \frac{1}{2} \overset{FT}{\longleftrightarrow} \pi\, \delta(\omega)$

Frequency shifting property states : $e^{j\beta t}\, x(t) \overset{FT}{\longleftrightarrow} X(\omega - \beta)$.

Then above equation will be, $\frac{1}{2} e^{j\omega_0 t} \overset{FT}{\longleftrightarrow} \pi\delta(\omega - \omega_0)$

and $\frac{1}{2} e^{-j\omega_0 t} \overset{FT}{\longleftrightarrow} \pi\delta(\omega + \omega_0)$ ... (Q.12.2)

Applying above results to equation (Q.12.1)

$$X(\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

Thus, $\boxed{\cos \omega_0 t \overset{FT}{\longleftrightarrow} \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]}$ .... (Q.12.3)

ii) $x(t) = \sin \omega_0 t$

$$= \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

Applying results of equation (Q.12.2) to above equation,

$$X(\omega) = \frac{\pi}{j}\delta(\omega - \omega_0) - \frac{\pi}{j}\delta(\omega + \omega_0)$$

Thus, $\boxed{\sin \omega_0 t \overset{FT}{\longleftrightarrow} \frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]}$ .... (Q.12.4)

Fig. Q.12.1 (a) shows magnitude plot of $\cos \omega_0 t$ and Fig. Q.12.1 (b) shows magnitude plot of $\sin \omega_0 t$.

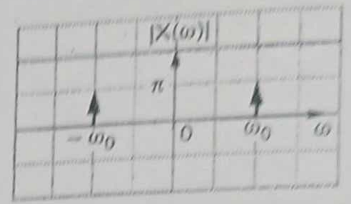


Fig. Q.12.1 (a) Spectrum of $\cos \omega_0 t$

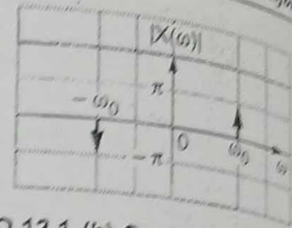


Fig. Q.12.1 (b) Spectrum of $\sin \omega_0 t$

Q.13 Using differentiation in frequency domain property, find Fourier transform of :
$y(t) = t\, x(t)$ where $x(t) = e^{-at}\, u(t)$. [SPPU : Dec.-13, Marks 5]

Ans. : $\underbrace{e^{-at} u(t)}_{x(t)} \overset{FT}{\longleftrightarrow} \underbrace{\frac{1}{a + j\omega}}_{X(\omega)}$

$\therefore \; -jt\; x(t) \overset{FT}{\longleftrightarrow} \frac{d}{d\omega} X(\omega)$ By frequency differentiation property

$\therefore \quad -jt \cdot e^{-at} u(t) \overset{FT}{\longleftrightarrow} \frac{d}{d\omega} \cdot \frac{1}{a + j\omega}$

$\therefore \quad t \cdot e^{-at} u(t) \longleftrightarrow j \frac{d}{d\omega} \cdot \frac{1}{a + j\omega}$

$$\longleftrightarrow j \cdot \frac{-j}{(a + j\omega)^2} = \frac{1}{(a + j\omega)^2}$$

4.4 : Interplay between Time and Frequency Domain using Sinc and Rectangular Signals

Important Points to Remember

- The rectangular signal represents low parts function and its fourier transform is a sinc pulse.
- The rectangular signal is time limited but its fourier transform, sinc pulse is not band limited.

- Rectangular signal and sinc function forms a fourier transform pair. i.e.,

  Rectangular signal $\xleftrightarrow{FT}$ Sinc function
- Similarly a sinc function in time domain have rectangular signal as fourier transform.
- Thus an infinite time function have bandlimited spectrum.

Q.14 Obtain the Fourier transform of a rectangular pulse and sinc pulses shown in Fig. Q.14.1. [SPPU : Dec.-11,17, May-17, Marks 8, May-15, Dec.-17,19, Marks 6, June-22, Marks 4]

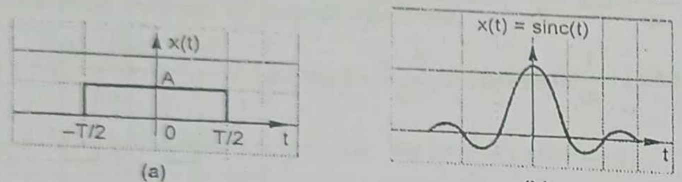


Fig. Q.14.1

OR Show that rectangular function in time domain to become sinc function in frequency domain. [SPPU : Dec.-13, Marks 3, May-14, Marks 6]

Ans. : i) $x(t)$ = A rect (t/T)

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \int_{-T/2}^{T/2} A\cdot e^{-j\omega t}\,dt = A\left[e^{-j\omega t}\right]_{-T/2}^{T/2}$$

$$= \frac{2A}{\omega}\sin\left(\frac{\omega T}{2}\right), \quad \text{Here} \quad \frac{e^{j\omega T/2}-e^{-j\omega T/2}}{2j} = \sin\left(\frac{\omega T}{2}\right)$$

Rearranging above equation,

$$X(\omega) = \frac{2A}{\omega}\cdot\frac{\sin\pi\cdot\frac{\omega T}{2\pi}}{\pi\cdot\frac{\omega T}{2\pi}}\cdot\frac{\omega T}{2}$$

$$= AT\cdot\frac{\sin\pi\cdot\left(\frac{\omega T}{2\pi}\right)}{\pi\cdot\left(\frac{\omega T}{2\pi}\right)} = AT\text{sinc}\frac{\omega T}{2\pi}, \quad \text{since sinc}\,\theta = \frac{\sin\pi\theta}{\pi\theta}$$

Thus, $$A\,rect\left(\frac{t}{T}\right) \xleftrightarrow{FT} \frac{2A}{\omega}\sin\left(\frac{\omega T}{2}\right) \text{ or } AT\text{ sinc}\frac{\omega T}{2\pi} \quad \text{.... (Q.14.1)}$$

Magnitude plot : $|X(\omega)| = AT\text{ sinc}\left(\frac{\omega T}{2\pi}\right) = \frac{2A}{\omega}\sin\frac{\omega T}{2}$

This function goes to zero at $\frac{\omega T}{2} = \pm\pi, \pm 2\pi, \pm 3\pi$, ... or $\omega = \pm\frac{2\pi}{T}, \pm\frac{4\pi}{T}, \pm\frac{6\pi}{T}$, ... By L' Hospital's rule,

$$|X(0)| = AT\text{ sinc}(0) = AT.$$

$$\angle X(\omega) = 0$$

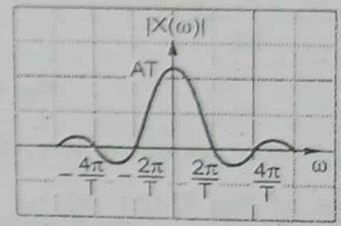


Fig. Q.14.2 Magnitude of 'rect' function

ii) $x(t)$ = sinc(t)

Consider equation (Q.14.1),

$$\underbrace{A\text{ rect}\left(\frac{t}{T}\right)}_{x(t)} \xleftrightarrow{FT} \underbrace{AT\text{ sinc}\left(\frac{\omega T}{2\pi}\right)}_{X(\omega)}$$

Apply duality property $X(t) \xleftrightarrow{FT} 2\pi\, x(-\omega)$ to above equation,

$$AT\text{ sinc}\left(\frac{tT}{2\pi}\right) \xleftrightarrow{FT} 2\pi\cdot A\text{ rect}\left(-\frac{\omega}{T}\right)$$

Here rect $(-x)$ = rect (x), since it is even function.

$$\therefore T\text{ sinc}\left(\frac{tT}{2\pi}\right) \xleftrightarrow{FT} 2\pi\text{ rect}\left(\frac{\omega}{T}\right)$$

Let $T = 2\pi$,

$$2\pi\text{ sinc}(t) \xleftrightarrow{FT} 2\pi\text{ rect}\left(\frac{\omega}{2\pi}\right)$$

$$\therefore \quad \text{sinc}(t) \xleftrightarrow{FT} \text{rect}\left(\frac{\omega}{2\pi}\right) \quad \text{.... (Q.14.2)}$$

Fig. Q.14.3 shows the magnitude plot.

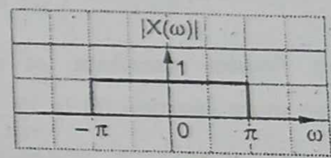


Fig. Q.14.3 rect $\left(\frac{\omega}{2\pi}\right)$

Q.15 Draw the amplitude spectrum of the following signals.

[SPPU : Dec.-06, Marks 10, Dec.-15, Marks 3]

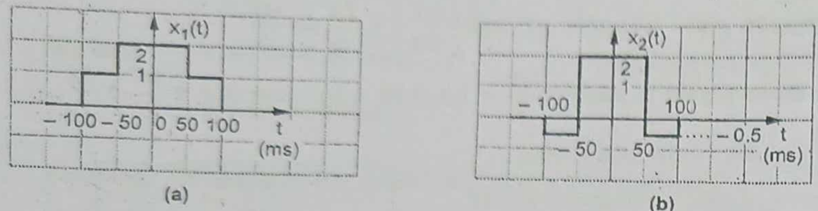


Fig. Q.15.1

Ans. : i) Fourier transform of $x_1(t)$

The pulse of Fig. Q.15.1 (a) can be synthesized as the sum of two pulses as shown in Fig. Q.15.2. Thus,

$$x_1(t) = x_{1a}(t) + x_{2a}(t)$$

$$= \text{rect}\left(\frac{t}{0.1}\right) + \text{rect}\left(\frac{t}{0.2}\right)$$

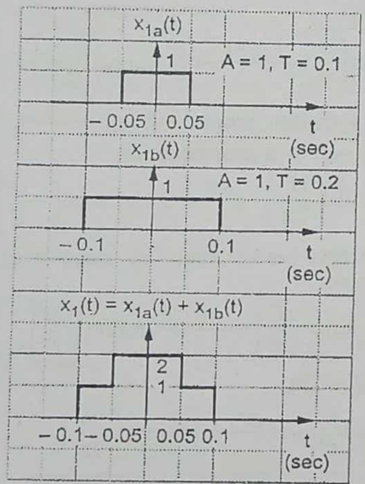


Fig. Q.15.2 Synthesis of $x_1(t)$

Taking Fourier transform of above equation using equation (Q.14.1).

$$X_1(\omega) = \frac{2}{\omega}\sin\left(\frac{0.1\omega}{2}\right) + \frac{2}{\omega}\sin\left(\frac{0.2\omega}{2}\right)$$

$$= \frac{2}{\omega}[\sin(0.05\omega) + \sin(0.1\omega)]$$

ii) Fourier transform of $x_2(t)$

From Fig. Q.15.3 we can write,

$$x_2(t) = x_{2a}(t) + x_{2b}(t)$$

$$= 2.5\,\text{rect}\left(\frac{t}{0.1}\right) + 0.5\,\text{rect}\left(\frac{t}{0.2}\right)$$

Taking Fourier transform of above equation using equation (Q.14.1),

$$X_2(\omega) = \frac{2(2.5)}{\omega}\sin\left(\frac{0.1\omega}{2}\right) + \frac{2(0.5)}{\omega}\sin\left(\frac{0.2\omega}{2}\right)$$

$$= \frac{5}{\omega}\sin(0.05\omega) + \frac{1}{\omega}\sin(0.1\omega)$$

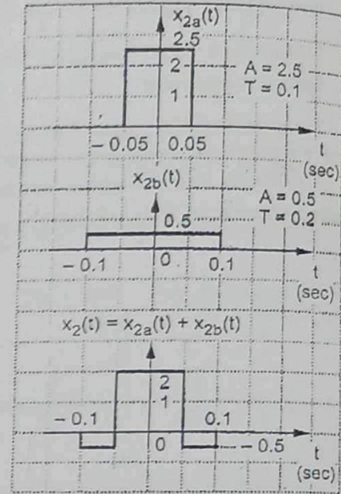


Fig. Q.15.3 Synthesis of $x_2(t)$

Q.16 Determine Fourier transform of the following signals :

i) $x(t) = \frac{d}{dt}e^{-2t\,u(t)}$ **ii)** $x(t) = \text{rect}\left(\frac{t}{2}\right)\cos\omega_0 t$.

[SPPU : May-16, Marks 6]

Ans. : i) $x(t) = \frac{d}{dt}e^{-2t}u(t)$

Let $x_1(t) = e^{-2t}u(t)$. Hence $X_1(\omega) = \frac{1}{2+j\omega}$. Consider the time differentiation property,

$$\frac{dx(t)}{dt} \xleftrightarrow{FT} j\omega X(\omega)$$

$$\therefore \quad x(t) = \frac{d}{dt}x_1(t) \xleftrightarrow{FT} j\omega X_1(\omega)$$

or

$$\frac{d}{dt}e^{-2t}u(t) \xleftrightarrow{FT} j\omega \cdot \frac{1}{2+j\omega} \quad \text{or} \quad X(\omega) = \frac{j\omega}{2+j\omega}$$

ii) $x(t) = rect\left(\frac{t}{2}\right) \cdot \cos \omega_0 t$

Let $x_1(t) = rect\left(\frac{t}{2}\right)$. Consider the standard fourier transform pair,

$$A\, rect\left(\frac{t}{T}\right) \xleftrightarrow{FT} AT \text{ sinc}\left(\frac{\omega T}{2\pi}\right)$$ with A = 1, T = 2,

$$rect\left(\frac{t}{2}\right) \xleftrightarrow{FT} 2\text{sinc}\left(\frac{\omega}{\pi}\right) = X_1(\omega)$$

Consider modulation theorem of fourier transform,

$$x(t)\cos(\omega_0 t + \phi) \xleftrightarrow{FT} \frac{e^{j\phi}}{2} X(\omega - \omega_0) + \frac{e^{-j\phi}}{2} X(\omega + \omega_0)$$

with $\phi = 0$, $x_1(t)\cos \omega_0 t \xleftrightarrow{FT} \frac{1}{2} X_1(\omega - \omega_0) + \frac{1}{2} X_1(\omega + \omega_0)$

$$\therefore rect\left(\frac{t}{2}\right)\cos \omega_0 t \xleftrightarrow{FT} \frac{1}{2} X_2 \text{ sin c}\left(\frac{\omega - \omega_0}{\pi}\right) + \frac{1}{2} X_2 \text{ sinc}\left(\frac{\omega + \omega_0}{\pi}\right)$$

$$\therefore X(\omega) = \text{sinc}\left(\frac{\omega - \omega_0}{\pi}\right) + \text{sinc}\left(\frac{\omega + \omega_0}{\pi}\right)$$

Q.17 Find Fourier transform of the following signals using appropriate properties :

(i) $x(t) = \frac{d}{dt}\{e^{-at}u(t)\}$

(ii) $x(t) = e^{-2t}u(t+2)$

☞ **[SPPU : May-18, June-22, Marks 6]**

Ans. :

(i) $x(t) = \frac{d}{dt}\{e^{-at}u(t)\}$

Let $x_1(t) = e^{-2t}u(t)$. Hence $X_1(\omega) = \frac{1}{2 + j\omega}$

Consider the time differentiation property,

$$\frac{dx(t)}{dt} \xleftrightarrow{FT} j\omega X(\omega)$$

$$\therefore \quad x(t) = \frac{d}{dt} x_1(t) \xleftrightarrow{FT} j\omega X_1(\omega)$$

or $\frac{d}{dt} e^{-2t}u(t) \xleftrightarrow{FT} j\omega \cdot \frac{1}{2 + j\omega}$ or $X(\omega) = \frac{j\omega}{2 + j\omega}$

(ii) Here $e^{-2t}u(t) = \xleftrightarrow{FT} \frac{1}{2 + j\omega}$

By time shift property, $e^{-2(t+2)}u(t+2) \xleftrightarrow{FT} e^{j2\omega} \frac{1}{2 + j\omega}$

$$\therefore \quad e^{-2t} \cdot e^{-4} u(t+2) \xleftrightarrow{FT} \frac{e^{j2\omega}}{2 + j\omega}$$

$$\therefore \quad e^{-2t}u(t+2) \xleftrightarrow{FT} \frac{e^4 \cdot e^{j2\omega}}{2 + j\omega}$$

Q.18 Find Fourier transform of the following signal $\frac{d}{dt}\{(e^{-3t}u(t) * e^{-3t}u(t-2))\}$

☞ **[SPPU : May-19, Marks 6]**

Ans. : Given :

$$x(t) = \frac{d}{dt}\{e^{-3t} \cdot u(t) * [e^{-3t} \cdot u(t-2)]\}$$

$$\therefore FT\{e^{-3t} \cdot u(t)\} = \frac{1}{3 + j\omega}$$

And,

$$\therefore FT\{e^{-3t} \cdot u(t-2)\} = FT\{e^{-3(t-2)} \cdot u(t-2) \cdot e^{-6}\}$$

$$= e^{-6} \cdot \frac{e^{-j2\omega}}{(3 + j\omega)}$$

$$\therefore FT\{e^{-3t}u(t) * e^{-3t} \cdot u(t-2)\} = \frac{e^{-6} \cdot e^{-j2\omega}}{(3 + j\omega)^2}$$ (By convolution theorem)

And, by differentiation property,

$$\therefore \quad FT\left\{\frac{d}{dt}[e^{-3t} \cdot u(t) * e^{-3t}u(t-2)]\right\} = \frac{j\omega \cdot e^{-6} \cdot e^{-j2\omega}}{(3 + j\omega)^2}$$

4.5 : Applications of Fourier Transform

Important Points to Remember

- The system function $H(\omega)$ is ratio of Fourier transforms of output to input. i.e.,

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$ This equation also gives frequency response.

- Impulse response of the system is obtained by taking inverse Fourier transform of $H(\omega)$ or system function.
- Fourier transform is used to solve differential equations.

Q.19 Determine frequency response and impulse response for the system described by the following differential equation. Assume zero initial condition.

$$\frac{dy(t)}{dt} + 3y(t) = x(t)$$

☞ [SPPU : Dec.-08, Marks 8, Dec.-16, Marks 6]

Ans. : i) Frequency response

Taking Fourier transform of given differential equation,

$$(j\omega)Y(\omega) + 3Y(\omega) = X(\omega)$$

$$\therefore \quad H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{3+j\omega}$$

$$\therefore \quad |H(\omega)| = \frac{1}{\sqrt{9+\omega^2}}$$

ii) Impulse response

$$H(\omega) = \frac{1}{3+j\omega}$$

Taking inverse Fourier transform,

$$h(t) = e^{-3t}u(t)$$

Q.20 The differential equation of the system is given as,

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6\,y(t) = -\frac{dx(t)}{dt}$$

Determine the frequency response and impulse response of this system.

Ans. : Taking Fourier transform of given differential equation,

$$(j\omega)^2 Y(\omega) + 5(j\omega)Y(\omega) + 6Y(\omega) = -j\omega X(\omega)$$

$$\therefore \quad Y(\omega)\left\{(j\omega)^2 + 5j\omega + 6\right\} = -j\omega X(\omega)$$

Hence system transfer function is,

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{-j\omega}{(j\omega)^2 + 5j\omega + 6} = \frac{-j\omega}{(j\omega+2)(j\omega+3)}$$

$$= \frac{2}{j\omega+2} - \frac{3}{j\omega+3} = 2\cdot\frac{1}{2+j\omega} - 3\cdot\frac{1}{3+j\omega}$$

Taking inverse Fourier transform,

$$h(t) = \left[2e^{-2t} - 3e^{-3t}\right]u(t)$$

Q.21 Find the transfer function and impulse response of the system describe by following differential equation.

$$\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 6y(t) = \frac{d}{dt}x(t) + x(t)$$

☞ [SPPU : Dec.-13, Marks 6]

Ans. : Taking Fourier transform of the given equation,

$$(j\omega)^2 Y(\omega) + 5(j\omega)Y(\omega) + 6Y(\omega) = (j\omega)X(\omega) + X(\omega)$$

$$\therefore \quad H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{j\omega+1}{(j\omega)^2 + 5(j\omega) + 6} \quad \text{(Transfer function)}$$

$$= \frac{j\omega+1}{(j\omega+3)(j\omega+2)} = \frac{2}{j\omega+3} - \frac{1}{j\omega+2}$$

Taking inverse Fourier transform,

$$h(t) = \left[2e^{-3t} - e^{-2t}\right]u(t)$$

Q.22 Find the transfer function of the following :

i) An ideal differentiator ii) An ideal integrator iii) An ideal delay of T second.

☞ [SPPU : May-15, Marks 6]

Ans. : i) Ideal differentiator :

Fig. Q.22.1 shows ideal differentiator. Its time domain equation is written as,

$$y(t) = \frac{d}{dt} x(t)$$

Fig. Q.22.1 System as differentiator

Taking Fourier transform,

$$Y(\omega) = j\omega \; X(\omega) \text{ using time differentiation properly.}$$

$$\therefore \quad H(\omega) = \frac{Y(\omega)}{X(\omega)} = j\omega$$

ii) Ideal integrator :

Fig. Q.22.2 shows ideal integrator. It's time domain equation can be written as

$$y(t) = \int x(t)\, dt$$

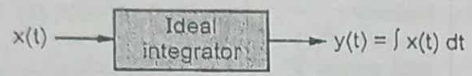


Fig. Q.22.2 System as integrator

Taking Fourier transform,

$$Y(\omega) = \frac{1}{j\omega} X(\omega) \text{ using integration properly and assuming}$$

$x(0) = 0$.

$$\therefore \quad H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{j\omega}$$

(iii) An ideal delay of 'T' second :

Mathematical equation for delay of 'T' second can be written as,

$$y(t) = x(t - T)$$

Taking Fourier transform of above equation using time shift properly,

$$Y(\omega) = e^{-j\omega T} \; X(\omega)$$

$$\therefore \quad H(\omega) = \frac{Y(\omega)}{X(\omega)} = e^{-j\omega T}$$

Q.23 Determine transfer function and impulse response of the system represented by :

$$\frac{d^2 y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 3\frac{dx(t)}{dt} + 2x(t), \quad \text{Assume zero initial}$$

conditions.

☞ [SPPU : Dec.-15, Marks 6]

Ans. : Taking Fourier transform of given differential equation,

$$(j\omega)^2\, Y(\omega) + 3(j\omega) Y(\omega) + 2Y(\omega) = 3\,(j\omega) X(\omega) + 2X(\omega)$$

$$\therefore \; Y(\omega)\,[(j\omega)^2 + 3(j\omega) + 2] = [3(j\omega) + 2]\, X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{3(j\omega)+2}{(j\omega)^2 + 3(j\omega) + 2} = \frac{4}{j\omega+2} - \frac{1}{j\omega+1}$$

Taking inverse Fourier transform, $h(t) = [4e^{-2t} - e^{-t}]\,u(t)$

FORMULAE AT A GLANCE

i) Fourier transform :

Fourier Transform : $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$ or

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t}\, dt \quad \dots (4.1)$$

Inverse Fourier Transform : $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t}\, df \quad \dots (4.2)$

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |X(f)|^2\, df = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega$$

ii) Discrete time Fourier transform :

DTFT : $X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \quad \dots (4.3)$

Inverse DTFT : $x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega \quad \dots (4.4)$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2\, d\omega$$

END...

Unit V

5 Laplace Transform

Laplace transform properties

Sr. No.	Name of the property	Property	ROC
1.	Linearity	$a_1 x_1(t) + a_2 x_2(t) \xleftrightarrow{L} a_1 X_1(s) + a_2 X_2(s)$	$R_1 \cap R_2$
2.	Time shifting	$x(t-t_0) \xleftrightarrow{L} e^{-st_0} X(s)$	R
3.	Shifting in s-domain	$e^{s_0 t} x(t) \xleftrightarrow{L} X(s-s_0)$	$R + Re(s_0)$
4.	Time scaling	$x(at) \xleftrightarrow{L} \frac{1}{\lvert a \rvert} X\left(\frac{s}{a}\right)$	$\frac{R}{a}$
5.	Differentiation in time domain	$\frac{d}{dt} x(t) \xleftrightarrow{L} sX(s)$ for bilateral LT $\frac{d}{dt} x(t) \xleftrightarrow{L} sX(s) - x(0-)$ for unilateral LT	R
6.	Differentiation in s-domain	$-tx(t) \xleftrightarrow{L} \frac{d}{ds} X(s)$	R
7.	Convolution	$x_1(t) * x_2(t) \xleftrightarrow{L} X_1(s) X_2(s)$	$R_1 \cap R_2$

Sr. No.	Name of the property	Property	ROC
8.	Integration in time domain	$\int_{-\infty}^{t} x(\tau) d\tau \xleftrightarrow{L} \frac{X(s)}{s}$ for bilateral LT $\int_{0-}^{t} x(\tau) d\tau \xleftrightarrow{L} \frac{X(s)}{s}$ for unilateral LT	$R \cap [Re(s) > 0]$
9.	Integration in s-domain	$\frac{x(t)}{t} \xleftrightarrow{L} \int_{s}^{\infty} X(s)\, ds$	R
10.	Periodic function $f(t)$ with period 'T' and first cycle denoted by $f_1(t)$	Let $f_1(t) \xleftrightarrow{L} F_1(s)$ then $f(t) \xleftrightarrow{L} \frac{1}{1-e^{-sT}} F_1(s)$	
11.	Initial value theorem	$x(0+) = \lim_{s \to \infty} s X(s)$	
12.	Final value theorem	$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$	

Table 5.1 Properties of bilateral and unilateral Laplace transform

Laplace transform pairs

Sr. No.	Time domain signal $x(t)$	Laplace transform $X(s)$	ROC
1.	$\delta(t)$	1	Entire s-plane
2.	$u(t)$	$\frac{1}{s}$	$Re(s) > 0$
3.	$r(t)$	$\frac{1}{s^2}$	$Re(s) > 0$
4.	$e^{at} u(t)$	$\frac{1}{s-a}$	$Re(s) > a$

5.	$-e^{at}\,u(-t)$	$\frac{1}{s-a}$	$Re(s) < a$
6.	$A\,\sin\omega_0 t\,u(t)$	$\frac{A\,\omega_0}{s^2+\omega_0^2}$	$Re(s) > 0$
7.	$A\,\cos\omega_0 t\,u(t)$	$\frac{A\,s}{s^2+\omega_0^2}$	$Re(s) > 0$
8.	$e^{-at}\sin\omega_0 t$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$Re(s) > -a$
9.	$e^{-at}\cos\omega_0 t$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$Re(s) > -a$
10.	$\sinh(at)$	$\frac{a}{s^2+a^2}$	$Re(s)>a$
11.	$\cosh(at)$	$\frac{s}{s^2+a^2}$	$Re(s)>a$
12.	$e^{-at}\sinh(bt)$	$\frac{b}{(s+a)^2-b^2}$	$Re(s)>(a+b)$
13.	$e^{-at}\cosh(bt)$	$\frac{s+a}{(s+a)^2-b^2}$	$Re(s)>(a+b)$
14.	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$Re(s) > 0$
15.	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$Re(s) < 0$
16.	$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$	$\frac{1}{(s+a)^n}$	$Re(s) > -a$
17.	$-\frac{t^{n-1}}{(n-1)!}e^{-at}u(-t)$	$\frac{1}{(s+a)^n}$	$Re(s) < -a$

Table 5.2 Commonly used Laplace transform pairs

5.1 : Definitions and Properties of Laplace Transform

Important Points to Remember

- Laplace transform and its inverse is given as,

$$X(s) = \int_{-\infty}^{\infty} x(t)\,e^{-st}\,dt$$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} X(s)e^{st}\,ds$$

Q.1 Derive the relationship between fourier transform and Laplace transform. [SPPU : Dec.-14,18, May-19, Marks 2]

Ans. : Laplace transform is given as,

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

since $s = \sigma + j\omega$,

$$X(s) = \int_{-\infty}^{\infty} x(t)\,e^{-(\sigma+j\omega)t}\,dt = \int_{-\infty}^{\infty} x(t)\,e^{-\sigma t}\,e^{-j\omega t}\,dt$$

$$= \underbrace{\int_{-\infty}^{\infty}\left\{x(t)e^{-\sigma t}\right\}e^{-j\omega t}\,dt}_{\text{Fourier transform}} \quad \text{... (Q.1.1)}$$

Thus, Laplace transform of $x(t)$ is basically the Fourier transform of $x(t)\,e^{-\sigma t}$. If $s = j\omega$, i.e. $\sigma = 0$, then above equation becomes,

$$X(s) = \int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt = X(j\omega)$$

Thus $X(s) = X(j\omega)$ when $s = j\omega$ ... (Q.1.2)

This means Laplace transform is same as Fourier transform when $s = j\omega$

Q.2 Explain the significance of region of convergence for Laplace transform. [SPPU : Dec.-08, Marks 2]

OR Define ROC and state its properties. [SPPU : Dec.-11, Marks 4]

Ans. : ROC : The range of values of 'σ' for which Laplace transform converges is called region of convergence, or RoC. The ROC has following properties :

1. No poles lie in ROC.
2. ROC of the causal signal is right hand sided. It is of the form Re(s) > a.
3. ROC of the noncausal signal is left hand sided. It is of the form Re(s) < a.
4. The shystem is stable if its ROC includeds $j\omega$ axis of s-plane.

Q.3 State the limitations of fourier transform and need of Laplace transform. [SPPU : June-22, Marks 6]

Ans. : Limitations of fourier transform

i) Fourier transform can be calculated only for the signals which are absolutely integrable. But Laplace transform exists for signals which are not absolutely integrable.

ii) Fourier transform is calculated only on the imaginary axis, but Laplace transform can be calculated over complete s-plane. Hence Laplace transform is more broader compared to Laplace transform.

iii) Laplace transform is more suitable for solving differential equations with initial conditions compared to Laplace transform.

Need of Laplace transform

- Laplace transform represents continuous time signals in terms of complex exponentials, i.e. e^{-st}.
- Continuous time systems are also analyzed more effectively using Laplace transforms. Laplace transform can be applied to the analysis of unstable systems also.

Q.4 State and prove periodic signal and time scaling properties of Laplace transform. [SPPU : May-14, Marks 6]

Ans. : i) Laplace transform of periodic signal

Let the Laplace transform of the first cycle of the periodic function be $X_1(s)$. Then the Laplace transform of the periodic function with period T is given as,

$$X(s) = \frac{1}{1-e^{-sT}} X_1(s) \quad \text{... (Q.4.1)}$$

Proof : Fig. Q.4.1 shows the periodic 'sine' wave. For this signal we can write,

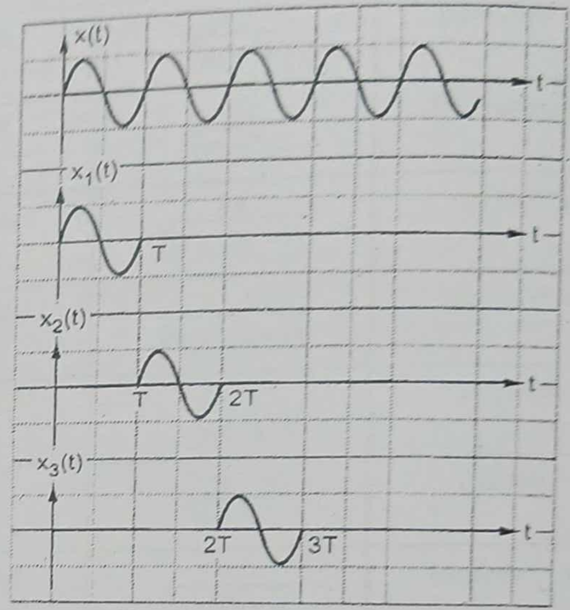


Fig. Q.4.1 Periodic signal

$$x(t) = x_1(t) + x_2(t) + x_3(t) + x_4(t) + .. \quad \text{... (Q.4.2)}$$

Note that $x_1(t)$, $x_2(t)$, $x_3(t)$ ...... is the same signal with shifts. i.e.,

$x_2(t) = x_1(t-T)$

$x_3(t) = x_1(t-2T)$

$x_4(t) = x_1(t-3T)$ ... and so on.

Hence equation (Q.4.2) can be written as,

$$x(t) = x_1(t) + x_1(t-T) + x_2(t-2T) + x_3(t-3T) + ...$$

We know that $x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s)$. Taking Laplace transform of above equation using time shifting property,

$$\mathcal{L}[x(t)] = X_1(s) + e^{-Ts} X_1(s) + e^{-2Ts} X_1(s) + e^{-3Ts} X_1(s) + e^{-4Ts} X_1(s) + ...$$

$$\therefore X(s) = X_1(s)\left\{1 + e^{-Ts} + e^{-2Ts} + e^{-4Ts} + ...\right\} = X_1(s) \cdot \frac{1}{1-e^{-Ts}}$$

ii) Time scaling

Statement : Expansion in time domain is equivalent to compression in frequency domain and vice versa.

$$x(at) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{s}{a}\right), \quad \text{ROC} : \frac{R}{a} \qquad \text{... (Q.4.3)}$$

Proof : $\mathcal{L}[x(at)] = \int_{-\infty}^{\infty} x(at)\, e^{-st}\, dt$

Let $at = \tau$, hence $t = \frac{\tau}{a}$, $dt = \frac{1}{a} d\tau$. The limits of integration will remain same.

$$\mathcal{L}[x(at)] = \int_{-\infty}^{\infty} x(\tau)\, e^{-s\frac{\tau}{a}} \frac{1}{a} d\tau = \frac{1}{a} \int_{-\infty}^{\infty} x(\tau)\, e^{-\left(\frac{s}{a}\right)\tau} d\tau$$

$$= \frac{1}{a} X\left(\frac{s}{a}\right) \quad \text{ROC} : \frac{R}{a} \qquad \text{... (Q.4.4)}$$

Similar procedure can be repeated for Laplace transform of $x(-at)$. We get,

$$\mathcal{L}[x(-at)] = \frac{1}{a} X\left(\frac{s}{-a}\right), \text{ROC} : \frac{R}{-a}$$

Above equation and equation (Q.4.4) can be combined as follows :

$$\mathcal{L}[x(at)] = \frac{1}{|a|} X\left(\frac{s}{a}\right), \quad \text{ROC} : \frac{R}{a}$$

As a special case with $a = -1$ we have,

$$x(-t) \overset{\mathcal{L}}{\longleftrightarrow} X(-s), \quad \text{ROC} : R \qquad \text{... (Q.4.5)}$$

This result shows that inverting the time axis inverts frequency axis as well as ROC.

Q.5 State and prove following properties of Laplace transform.

i) Time shifting

[SPPU : May-07, Marks 3, May-12, Marks 5, Dec.-19, June-22, Marks 2]

ii) Frequency shifting

[SPPU : Dec.-07, Marks 6, Dec.-11, Marks 4, June-22, Marks 2]

Ans. : i) Time shifting (Translation in time domain)

Statement : A time shift in the signal introduces frequency shift in frequency domain.

$$\mathcal{L}[x(t-t_0)] = e^{-st_0} X(s) \quad \text{ROC} : R \qquad \text{... (Q.5.1)}$$

Proof : $\mathcal{L}[x(t-t_0)] = \int_{-\infty}^{\infty} x(t-t_0)\, e^{-st}\, dt$

Let $\tau = t - t_0$, hence $t = \tau + t_0$ and $d\tau = dt$. The limits of integration will be $-\infty$ to ∞.

$$\therefore \mathcal{L}[x(t-t_0)] = \int_{-\infty}^{\infty} x(\tau)\, e^{-s(\tau+t_0)}\, d\tau = \int_{-\infty}^{\infty} x(\tau)\, e^{-s\tau} e^{-st_0}\, d\tau$$

$$= e^{-st_0} \underbrace{\int_{-\infty}^{\infty} x(\tau)\, e^{-s\tau}\, d\tau}_{X(s)} = e^{-st_0} X(s), \quad \text{ROC} : R$$

ii) Shifting in s-Domain (Complex translation) or Frequency shifting

Statement : A shift in the frequency domain is equivalent to multiplying the time domain signal by complex exponential.

$$\mathcal{L}\left[e^{s_0 t} x(t)\right] = X(s-s_0), \quad \text{ROC} : R + Re(s_0) \qquad \text{... (Q.5.2)}$$

Proof : $\mathcal{L}\left[e^{s_0 t} x(t)\right] = \int_{-\infty}^{\infty} e^{s_0 t} x(t) e^{-st}\, dt = \int_{-\infty}^{\infty} x(t)\, e^{-(s-s_0)t}\, dt$

$$= X(s-s_0) \text{ with ROC} : R + Re(s_0)$$

Q.6 State and prove differentiation property of Laplace transform.

[SPPU : May-05, Marks 3, May-06, Marks 6, May-08, Dec.-19, June-22, Marks 2, Dec.-08, 11, Marks 4, Dec.-12, Marks 5]

Ans. : i) Differentiation in time domain

Statement : Differentiation in time domain adds a zero to the system.

$$\frac{d}{dt}x(t) \xleftrightarrow{\mathcal{L}} sX(s), \quad \text{ROC : R} \qquad \ldots (Q.6.1)$$

Proof : $x(t) = \frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} X(s)\, e^{st}\, ds$ (Inverse Laplace transform)

Differentiate both sides of above equation with respect to 't' i.e.,

$$\frac{d}{dt}x(t) = \frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} X(s)\, s e^{st}\, ds = \underbrace{\frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} [sX(s)]\, e^{st}\, ds}_{\text{Inverse Laplace of } sX(s)}$$

For multiple order derivative.

$$\frac{d^n}{dt^n}x(t) \xleftrightarrow{\mathcal{L}} s^n X(s), \quad \text{ROC containing R} \qquad \ldots (Q.6.2)$$

ii) Differentiation in s-domain

Statement : Differentiation in s-domain corresponds to multiplying the time domain sequence by $-t$.

$$-t\,x(t) \xleftrightarrow{\mathcal{L}} \frac{d}{ds}X(s), \quad \text{ROC : R} \qquad \ldots (Q.6.3)$$

Proof : $X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$

Differentiating above equation with respect to 's',

$$\frac{d}{ds}X(s) = \int_{-\infty}^{\infty} x(t)(-t)\, e^{-st}\, dt = \underbrace{\int_{-\infty}^{\infty} [-t\,x(t)]\, e^{-st}\, dt}_{\text{Laplace transform of } -t\,x(t)}$$

$$\therefore \; -t\,x(t) \xleftrightarrow{\mathcal{L}} \frac{d}{ds}X(s)$$

For multiple order differentiation in s-domain,

$$(-t)^n x(t) \xleftrightarrow{\mathcal{L}} \frac{d^n}{ds^n}X(s), \quad \text{ROC : R} \qquad \ldots (Q.6.4)$$

Q.7 State and prove convolution property of Laplace transform.

[SPPU : May-07, Marks 3, Dec.-08, Marks 4, Dec.-16, Marks 6, Dec.-19, June-22, Marks 2]

Ans. : Statement : The Laplace transform of convolution of two functions is equivalent to multiplication of their Laplace transforms.

$$x_1(t) * x_2(t) \xleftrightarrow{\mathcal{L}} X_1(s)\, X_2(s), \quad \text{ROC : containing } R_1 \cap R_2 \qquad \ldots (Q.7.1)$$

Proof : $x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$ (convolution)

Taking Laplace transform of both the sides,

$$\mathcal{L}[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau\right\} e^{-st}\, dt$$

Changing the order of integration,

$$\mathcal{L}[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} x_1(\tau)\, d\tau \int_{-\infty}^{\infty} x_2(t-\tau)\, e^{-st}\, dt$$

Let $t-\tau = \lambda$ in second integration. Hence $t = \lambda + \tau$ and $dt = d\lambda$. The integration limits will remain same.

$$\therefore \mathcal{L}[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} x_1(\tau)\, d\tau \int_{-\infty}^{\infty} x_2(\lambda)\, e^{-s(\lambda+\tau)}\, d\lambda$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, d\tau \int_{-\infty}^{\infty} x_2(\lambda)\, e^{-s\lambda} \cdot e^{-s\tau}\, d\lambda$$

$$= \underbrace{\int_{-\infty}^{\infty} x_1(\tau) e^{-s\tau}\, d\tau}_{X_1(s)} \; \underbrace{\int_{-\infty}^{\infty} x_2(\lambda) e^{-s\lambda}\, d\lambda}_{X_2(s)}$$

$$= X_1(s)\, X_2(s), \quad \text{ROC : } R_1 \cap R_2$$

Q.8 State and prove the integration property of Laplace transform.

[SPPU : May-05, Marks 3, May-08, June-22, Marks 2, May-10, Marks 6]

Ans. : i) Integration in time domain

Statement : Time domain integration adds a pole to the system.

$$\int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{\mathcal{L}} \frac{X(s)}{s}, \quad \text{ROC} : R \cap [Re(s) > 0] \qquad \dots \text{(Q.8.1)}$$

Proof : $x(t) * u(t) = \int_{-\infty}^{\infty} u(t-\tau) x(\tau)\, d\tau$

$u(t-\tau) = 1$ for $\tau \le t$. Hence above equation becomes,

$$x(t) * u(t) = \int_{-\infty}^{t} 1\; x(\tau)\, d\tau = \int_{-\infty}^{t} x(\tau)\, d\tau$$

i.e. $\int_{-\infty}^{t} x(\tau)\, d\tau = x(t) * u(t)$

Taking Laplace transform of both sides,

$$\mathcal{L}\left\{\int_{-\infty}^{t} x(\tau)\, d\tau\right\} = \mathcal{L}\{x(t) * u(t)\} \quad x(t) \xleftrightarrow{\mathcal{L}} X(s) \text{ and } u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s}$$

$$= X(s)\frac{1}{s}, \text{ By convolution property}$$

$$= \frac{X(s)}{s}, \quad \text{ROC} : R \cap [Re(s) > 0]$$

For multiple order of integration,

$$\mathcal{L}\left\{\int_{-\infty}^{t}\int_{-\infty}^{t}\dots\int_{-\infty}^{t} x(t)\, dt_1\, dt_2 \dots dt_n\right\} = \frac{X(s)}{s^n} \qquad \dots \text{(Q.8.2)}$$

ii) Integration in s-domain

Statement : Frequency domain integration corresponds to dividing the time domain signal by t.

$$\frac{x(t)}{t} \xleftrightarrow{\mathcal{L}} \int_{s}^{\infty} X(s)\, ds, \quad \text{ROC} : R \qquad \dots \text{(Q.8.3)}$$

Proof : $\int_{s}^{\infty} X(s)\, ds = \int_{s}^{\infty}\left[\int_{-\infty}^{\infty} x(t) e^{-st}\, dt\right] ds$,

Changing the order of integration and rearranging the terms,

$$\int_{s}^{\infty} X(s)\, ds = \int_{-\infty}^{\infty} x(t)\left[\int_{s}^{\infty} e^{-st}\, ds\right] dt = \int_{-\infty}^{\infty} x(t)\left[\frac{e^{-st}}{-t}\right]_{s}^{\infty} dt$$

$$= \int_{-\infty}^{\infty} x(t)\left[\frac{e^{-\infty} - e^{-st}}{-t}\right] dt = \int_{-\infty}^{\infty} x(t)\frac{e^{-st}}{t}\, dt, \text{ ROC : R}$$

$$= \int_{-\infty}^{\infty} \frac{x(t)}{t} e^{-st}\, dt = \mathcal{L}\left[\frac{x(t)}{t}\right], \quad \text{ROC : R}$$

Q.9 Calculate the Laplace transform of following functions and plot their ROC.

i) x(t) = e^{at}**u(t) ii) x(t) =** $-e^{at} u(-t)$ [SPPU : Dec.-16, Marks 6]

Ans. : i) $x(t) = e^{at} u(t)$

$$X(s) = \int_{-\infty}^{\infty} e^{at} u(t) e^{-st}\, dt \text{ By definition of Laplace transform}$$

$$= \int_{0}^{\infty} e^{at} e^{-st}\, dt, \qquad \text{Since u(t) = 1 for } 0 \le t \le \infty$$

$$= \int_{0}^{\infty} e^{-(s-a)t}\, dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_{0}^{\infty} = \lim_{t\to\infty}\left[\frac{e^{-(s-a)t}}{-(s-a)}\right] - \lim_{t\to 0}\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]$$

Here $\lim_{t\to\infty}\left[\frac{e^{-(s-a)t}}{-(s-a)}\right] = \frac{e^{-\infty}}{-(s-a)} = 0$ if $(s-a) > 0$

$$\therefore \quad X(s) = 0 - \left[\frac{e^{0}}{-(s-a)}\right] \text{ for } (s-a) > 0 = \frac{1}{s-a} \text{ for } s > a.$$

The shaded area in Fig Q.9.1 is called the Region Of Convergence (ROC) of Laplace transform.

Thus,

$$e^{at}\,u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s-a},\ \text{ROC}: s>a \quad \dots \text{(Q.9.1)}$$

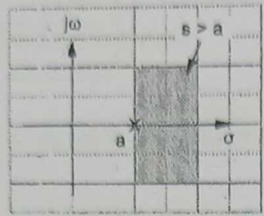


Fig. Q.9.1 ROC for $e^{at}\,u(t)$

*Some times above ROC is also written as Re(s)>a, since Re(s) is real part of 's' i.e. σ Hence **ROC** : σ > a or Re(s) > a.*

ii) $x(t) = -e^{at}\,u(-t)$

$$X(s) = \int_{-\infty}^{\infty} -e^{at}\,u(-t)\,e^{-st}\,dt = \int_{-\infty}^{0} -e^{at}\,e^{-st}\,dt, \text{ Since } u(-t) = 1 \text{ for}$$

$-\infty \le t \le 0$

$$= -\int_{-\infty}^{0} e^{-(s-a)t}\,dt = \left[\frac{e^{-(s-a)t}}{s-a}\right]_{-\infty}^{0} = \lim_{t\to 0}\left[\frac{e^{-(s-a)t}}{s-a}\right] - \lim_{t\to-\infty}\left[\frac{e^{-(s-a)t}}{s-a}\right]$$

Here,

$$\lim_{t\to-\infty}\left[\frac{e^{-(s-a)t}}{s-a}\right] = \frac{e^{-\infty}}{s-a} = 0 \text{ if } s<a$$

$$\therefore \quad X(s) = \frac{e^{0}}{s-a} - 0 \text{ if } s<a$$

$$= \frac{1}{s-a},\ \text{ROC}: s<a$$

The shaded region in Fig. Q.9.2 shows the ROC of s < a.

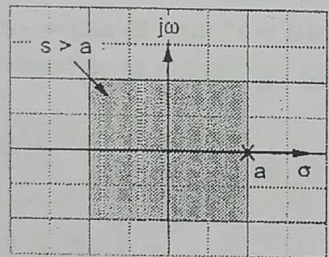


Fig. Q.9.2 ROC for $-e^{at}\,u(-t)$

Thus,

$$-e^{at}\,u(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{s-a},\ \text{ROC } s<a \quad \dots \text{(Q.9.2)}$$

Q.10 Obtain the Laplace transforms and ROC of following signals :
i) x(t) = u(t) ii) x(t) = δ(t) iii) x(t) = r(t) iv) x(t) = $te^{-at}\,u(t)$

☞ [SPPU : May-19, Dec.-19, Marks 4]

Ans. : i) $x(t) = u(t)$

$$\mathcal{L}\,[u(t)] = \int_{-\infty}^{\infty} u(t)\,e^{-st}\,dt = \int_{0}^{\infty} e^{-st}\,dt, \text{ since } u(t) = 1 \text{ for } 0 \le t \le \infty$$

$$= \left[\frac{e^{-st}}{-s}\right]_{0}^{\infty} = \frac{e^{-s\times\infty} - e^{0}}{-s}$$

Here $e^{-s\times\infty} = e^{-\infty} = 0$ if s > 0. Then above equation will be,

$$\mathcal{L}\,[u(t)] = \frac{1}{s},\ \text{ROC}: s>0 \text{ or } \sigma>0$$

Thus,

$$u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s},\ \text{ROC}: s>0 \text{ or } \sigma>0 \quad \dots \text{(Q.10.1)}$$

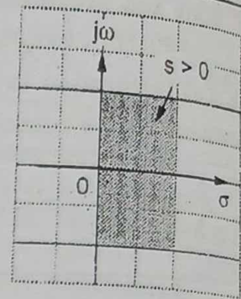


Fig. Q.10.1 ROC

Fig. Q.10.1 show the ROC.

ii) $x(t) = \delta(t)$

$$\mathcal{L}\,[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)\,e^{-st}\,dt$$

Here use $\int_{-\infty}^{\infty} \delta(t-t_0)\,x(t)\,dt = x(t_0)$

with $t_0 = 0$, above equation becomes,

$$\mathcal{L}\,[\delta(t)] = e^{0} = 1.$$

This is convergent for all vales of s.

$$\therefore \quad \delta(t) \xleftrightarrow{\mathcal{L}} 1,\ \textbf{ROC}: \text{Entire s-plane} \quad \dots \text{(Q.10.2)}$$

iii) $x(t) = r(t)$

$$\mathcal{L}\,[r(t)] = \int_{-\infty}^{\infty} r(t)\,e^{-st}\,dt = \int_{-\infty}^{\infty} t\,u(t)\,e^{-st}\,dt, \text{ since } r(t) = t\,u(t)$$

$$= \int_{0}^{\infty} t\,e^{-st}\,dt$$

Integrating by parts,

$$\mathcal{L}\,[r(t)] = \left[t\cdot\frac{e^{-st}}{-s}\right]_{0}^{\infty} - \int_{0}^{\infty} 1\cdot\frac{e^{-st}}{-s}\,dt$$

Here, $\dfrac{te^{-st}}{-s} = \dfrac{\infty\cdot e^{-s\times\infty} - 0}{-s} = 0$ if s > 0

$$\therefore \quad \mathcal{L}\,[r(t)] = 0 - \int_0^{\infty} \frac{e^{-st}}{-s}\, dt = \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^{\infty}$$

$$= \frac{1}{s} \cdot \frac{e^{-s\times\infty} - e^0}{-s} = \frac{1}{s^2} \text{ if } s > 0$$

$$\therefore \quad \boxed{r(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s^2}, \text{ROC} : s>0 \text{ or } \sigma > 0} \quad \text{... (Q.10.3)}$$

Similarly, $\frac{t^{n-1}}{(n-1)!} \xleftrightarrow{\mathcal{L}} \frac{1}{s^n}$, ROC : $Re\,(s) > 0$ or $\sigma > 0$

iv) $x(t) = te^{-at}\, u(t)$

We know that, $\underbrace{e^{-at}\, u(t)}_{x_1(t)} \xleftrightarrow{\mathcal{L}} \underbrace{\frac{1}{s+a}}_{X_1(s)}$, ROC : $s > -a$ or $Re\,(s) > -a$

By differentiation in s-domain property,

$$-t\, x_1(t) \xleftrightarrow{\mathcal{L}} \frac{d}{ds}, X_1(s)$$

$$-t \cdot e^{-at}\, u(t) \xleftrightarrow{\mathcal{L}} \frac{d}{ds} \cdot \frac{1}{s+a}$$

$$\therefore \quad \boxed{te^{-at}\, u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+a)^2}, \text{ROC} : Re(s) \text{ or } \sigma > -a} \quad \text{... (Q.10.4)}$$

Similarly, $-t\,[t\, e^{-at}\, u(t)] \xleftrightarrow{\mathcal{L}} \frac{d}{ds} \cdot \frac{1}{(s+a)^2}$

$$\therefore \quad \frac{t^2}{2} e^{-at}\, u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+a)^3} \quad \text{... (Q.10.5)}$$

$$\therefore \quad \frac{t^{n-1}}{(n-1)!}\, e^{-at}\, u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+a)^n} \quad \text{... (Q.10.6)}$$

Q.11 Obtain Laplace transforms of following signals :
i) x(t) = A sin ω_0t u(t) ii) x(t) = A cos ω_0t u(t).

[SPPU : May-09, Dec.-19, Marks 2]

Ans. : i) $x(t) = A \sin \omega_0 t\, u(t)$

$$X(s) = \int_{-\infty}^{\infty} A \sin \omega_0 t\, u(t)\, e^{-st}\, dt$$

$$= A \int_{-\infty}^{\infty} \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\, u(t)\, e^{-st}\, dt, \text{ since } \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$= \frac{A}{2j}\left[\int_{-\infty}^{\infty} e^{j\omega_0 t}\, u(t)\, e^{-st}\, dt - \int_{-\infty}^{\infty} e^{-j\omega_0 t}\, u(t)\, e^{-st}\, dt\right]$$

$$= \frac{A}{2j}\left[\mathcal{L}\,[e^{j\omega_0 t}\, u(t)] - \mathcal{L}\,[e^{-j\omega_0 t}\, u(t)]\right]$$

We know that $e^{at}\, u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s-a}$, ROC : $s > a$.

Then above equation can be written as,

$$X(s) = \frac{A}{2j}\left\{\frac{1}{s - j\omega_0} - \frac{1}{s + j\omega_0}\right\} \text{ ROC : } s > j\omega \text{ and } s > -j\omega.$$

Here $s > j\omega$ can be written as $\sigma + j\omega > 0 + j\omega$, hence $\sigma > 0$. Therefore ROC will be $Re(s)$ or $\sigma > 0$.

$$X(s) = \frac{A}{2j}\left\{\frac{2j\omega_0}{s^2 + \omega^2}\right\} = \frac{A\omega_0}{s^2 + \omega^2}$$

$$\therefore \quad \boxed{A \sin \omega_0\, t\, u(t) \xleftrightarrow{\mathcal{L}} \frac{A\omega_0}{s^2 + \omega^2}, \text{ROC} : \sigma > 0} \quad \text{... (Q.11.1)}$$

ii) $X(t) = A \cos \omega_0 t\, u(t)$

$$X(s) = \int_{-\infty}^{\infty} A \cos \omega_0 t\, u(t)\, e^{-st}\, dt$$

$$= A \int_{-\infty}^{\infty} \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} \cdot u(t)\, e^{-st}\, dt, \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$= \frac{A}{2}\left[\int_{-\infty}^{\infty} e^{j\omega_0 t}\, u(t)\, e^{-st}\, dt + \int_{-\infty}^{\infty} e^{-j\omega_0 t}\, u(t)\, e^{-st}\, dt\right]$$

$$= \frac{A}{2}\left\{\mathcal{L}\,[e^{j\omega_0 t}\, u(t)] + \mathcal{L}\,[e^{-j\omega_0 t}\, u(t)]\right\}$$

$$= \frac{A}{2}\left\{\frac{1}{s-j\omega_0}+\frac{1}{s+j\omega_0}\right\}, \text{ ROC : } s > j\omega \text{ and } s > -j\omega \text{ i.e. } Re(s) \text{ or } \sigma > 0$$

$$= \frac{A\cdot s}{s^2+\omega_0^2}$$

$$\therefore \boxed{A\cos\omega_0 t\, u(t) \xleftrightarrow{\mathcal{L}} \frac{A\cdot s}{s^2+\omega_0^2}, \text{ ROC} : \sigma > 0} \quad \text{... (Q.11.2)}$$

Q.12 Obtain Laplace transform of damped sine and cosine wave :
(i) $x_1(t) = e^{-at}\sin\omega t$ (ii) $x_2(t) = e^{-at}\cos\omega t$

☞ [SPPU : Dec.-07, May-17, Marks 4]

Ans. : (i) $x_1(t) = e^{-at}\sin\omega t$

Here $x_1(t) = e^{-at}\cdot\frac{e^{j\omega t}-e^{-j\omega t}}{2j}$, since $\sin\theta \frac{e^{j\theta}-e^{-j\theta}}{2j}$

$$= \frac{1}{2j}\{e^{-(a-j\omega)t}-e^{-(a+j\omega)t}\}$$

$$\therefore X_1(s) = \frac{1}{2j}\left\{\frac{1}{s+(a-j\omega)}+\frac{1}{s+(a+j\omega)}\right\}, \text{ ROC : } \sigma > -a$$

$$= \frac{\omega}{(s+a)^2+\omega^2}$$

Thus, $$\boxed{e^{-at}\sin\omega t \xleftrightarrow{\mathcal{L}} \frac{\omega}{(s+a)^2+\omega^2}, \text{ ROC : } \sigma > -a} \quad \text{...(Q.12.1)}$$

(ii) $x_2(t) = e^{-at}\cos\omega t$

Here $x_2(t) = e^{-at}\cdot\frac{e^{j\omega t}+e^{-j\omega t}}{2}$, since $\cos\theta \frac{e^{j\theta}+e^{-j\theta}}{2}$

$$= \frac{1}{2}\{e^{-(a-j\omega)t}-e^{-(a+j\omega)t}\}$$

$$\therefore X_2(s) = \frac{1}{2}\left\{\frac{1}{s+(a-j\omega)}+\frac{1}{s+(a+j\omega)}\right\}, \text{ ROC : } \sigma > -a$$

$$= \frac{s+a}{(s+a)^2+\omega^2}$$

Thus, $e^{-at}\cos\omega t \xleftrightarrow{\mathcal{L}} \frac{s+a}{(s+a)^2+\omega^2}$, ROC : $\sigma > -a$

Q.13 A signal x(t) has Laplace transform :

$$X(s) = \frac{s+2}{s^2+4s+5}$$

Find the Laplace transform of the following signals

1) $y_1(t) = t\,x(t)$ 2) $y_2(t) = e^{-t}x(t)$ ☞ [SPPU : Dec.-18, Marks 6]

Ans. :

Given : $X(s) = \frac{s+2}{s^2+4s+5}$

1) $Y_1(s) = \mathcal{L}[y_1(t)] = \mathcal{L}[t\cdot x(t)]$

Using differentiation in s-domain property,

$$\boxed{\mathcal{L}\{t\cdot x(t)\} = -\frac{d}{ds}X(s) = \frac{s^2+4s+3}{(s^2+4s+5)^2}}$$

2) Using frequency shifting property,

$$\mathcal{L}\{e^{-at}\cdot u(t)\} = X(s+a)$$

$$\therefore \boxed{\mathcal{L}[e^{-at}\cdot x(t)] = X(s+1) = \frac{s+3}{(s+1)^2+4(s+1)+5}}$$

Q.14 Find the Laplace transform of the following signal and sketch ROC : $x(t) = e^{-3t}u(t)+e^{-5t}u(t)$ ☞ [SPPU : Dec.-18,22, May-19, Marks 6]

Ans. :

Given : $x(t) = e^{-3t}\cdot u(t)+e^{-5t}\cdot u(t)$

Taking LT on both sides,

$$X(s) = \frac{1}{s+3}+\frac{1}{s+5}\ ; \text{ ROC } Re\{s\} > -3$$

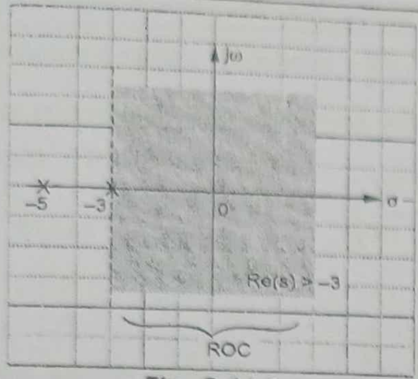


Fig. Q.14.1

Q.15 Find the Laplace transform of given periodic signal :

[SPPU : Dec.-17, Marks 6]

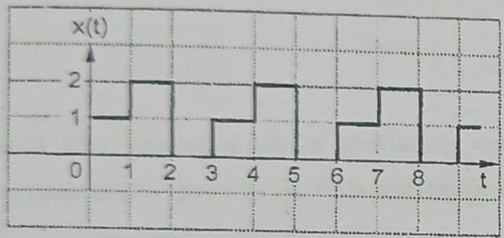


Fig. Q.15.1

Ans. : One pulse of $x(t)$ can be represented as addition of two unit step functions. i.e.,

$$x_1(t) = u(t) + u(t-1) - 2u(t-2)$$

Taking Laplace transform,

$$X_1(s) = \frac{1}{s} + \frac{e^{-s}}{s} - \frac{2e^{-2s}}{s} = \frac{1+e^{-s}-2e^{-2s}}{s}$$

This pulse is periodic with period T = 3. Then Laplace transform of periodic signal is given as,

$$X(s) = \frac{1}{1-e^{sT}} X_1(s) = \frac{1}{1-e^{-3s}} \cdot \frac{1+e^{-s}-2e^{-2s}}{s}$$

$$= \frac{1+e^{-s}-2e^{-2s}}{s(1-e^{-3s})}$$

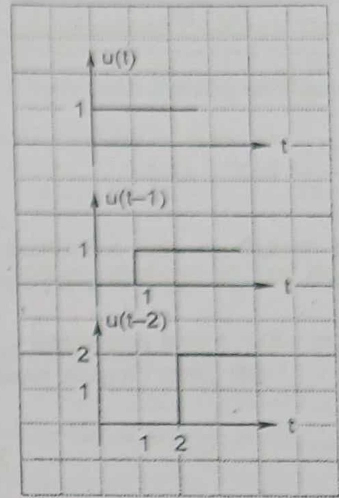


Fig. Q.15.2 : $x_1(t)$ synthesis

Q.16 Find the Laplace transform of the following with ROC :

i) $x(t) = u(t-5)$ [SPPU : Dec.-05, 08, Marks 4 , May-09, 10, Marks 2]

ii) $x(t) = e^{5t} u(-t+3)$ [SPPU : May-08, Marks 8, May-17, Marks 3]

Ans. : i) $x(t) = u(t-5)$

$u(t-5) \xleftrightarrow{\mathcal{L}} e^{-5s}\frac{1}{s}$, **ROC** : s > 0 By time shifting property

ii) $x(t) = e^{-5t} u(-t+3)]$

$$\therefore \quad X(s) = \int_{-\infty}^{\infty} e^{5t} u(-t+3)^{-st} dt = \int_{-\infty}^{\infty} e^{5t} e^{-st} dt$$

since $u(-t+3) = 1$ for $-\infty \le t \le 3$

$$= \int_{-\infty}^{3} e^{-s(s-5)} dt = \left[\frac{e^{-(s-5)t}}{-(s-5)}\right]_{-\infty}^{3} = \frac{e^{-(s-5)3}}{-(s-5)} + \frac{e^{-(s-5)(\infty)}}{-(s-5)}$$

Here $e^{-(s-5)(-\infty)} = e^{-\infty} = 0$, if $(s-5) < 0$. Thus the second term of above equation becomes zero.

$$\therefore \quad X(s) = \frac{e^{-(s-5)3}}{-(s-5)} \text{ for } (s-5)<0$$

$$= \frac{e^{-3s+15}}{-(s-5)} \quad \text{ROC} : s<5$$

Q.17 Find the Laplace transform and associated ROC for each of the following signals : i) $\delta(t-t_0)$ ii) $e^{-2t}[u(t)-u(t-5)]$

iii) $\sum_{k=0}^{\infty}\delta(t-kT)$ iv) $\delta(at+b)$ a, b real constants.

☞ [SPPU : May-10, Marks 8, Dec.-17, Marks 7]

Ans. : i) $x(t) = \delta(t-t_0)$

$\delta(t) \xleftrightarrow{\mathcal{L}} 1$ ROC : Entire s-plane

$\therefore \quad \delta(t-t_0) \xleftrightarrow{\mathcal{L}} 1-e^{-st_0}$ By time shifting property

$\therefore \quad X(\delta) \xleftrightarrow{\mathcal{L}} e^{-st_0}$ ROC : Entire s-plane

ii) $x_2(t) = e^{-2t}[u(t)-u(t-5)]$

$$= e^{-2t}u(t)-e^{-2t}u(t-5)$$

Let us rearrange this equation as,

$$X_3(t) = e^{-2t}u(t)-e^{-2(t-5+5)}u(t-5)$$

$$= e^{-2t}u(t)-e^{-2(t-5)}.e^{-10}u(t-5)$$

$$= e^{-2t}u(t)-e^{-10}.e^{-2(t-5)}u(t-5)$$

Here use $e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}$ and

time shift property : $x(t-t_0) \xleftrightarrow{\mathcal{L}} e^{-st_0}X(s)$.

$$\therefore \quad X_3(s) = \frac{1}{s+2}e^{-10}.e^{-5}\frac{1}{s+2} = \frac{1-e^{-5(s+2)}}{s+2}$$

iii) $x_3(t) = \sum_{k=0}^{\infty}\delta(t-kT)$

We have $\delta(t) \xleftrightarrow{\mathcal{L}} 1$ and time shifting property,

$\delta(t-kT) \xleftrightarrow{\mathcal{L}} e^{-kT_s}$

$$\therefore \quad X_4(s) = \sum_{k=0}^{\infty}e^{-kT_s}$$

iv) $x_4(t) = \delta(at+b)$

Here $x_4(t) = \delta\left[a\left(t+\frac{b}{a}\right)\right]$

$$\delta(t) \xleftrightarrow{\mathcal{L}} 1$$

$\delta\left(t+\frac{b}{a}\right) \xleftrightarrow{\mathcal{L}} e^{\frac{b}{a^s}}$, By time shifting property

$\delta\left[a\left(t+\frac{b}{a}\right)\right] \xleftrightarrow{\mathcal{L}} \frac{1}{a}e^{\left(\frac{b}{a}\right)\cdot\frac{s}{a}}$, By time scaling property

$$\therefore \quad \delta(at+b) \xleftrightarrow{\mathcal{L}} \frac{1}{a}^{bs/a^2}$$

Q.18 Find the Laplace transform of the following with ROC :
(i) $x(t) = \frac{d}{dt}te^{-t}u(t)$ ii) $x(t) = e^{-3t}u(t)*\cos(t-2)u(t-2)$.

☞ [SPPU : May-18, Marks 4]

Ans. : i) $e^{-t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+1}$

By differentiation in s - domain, $te^{-t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+1)^2}$

By differentiation in domain, $\frac{d}{dt}[te^{-t}u(t)] \xleftrightarrow{\mathcal{L}} \frac{1}{(s+1)^2}$

ii) Here $e^{-3t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+3}$ and cost $t \cdot u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s^2+1}$

By time shifting property, $\cos(t-2)u(t-2) \xleftrightarrow{\mathcal{L}} e^{-2s}.\frac{s}{s^2+1}$

By convolution in time,

$$e^{-3t}u(t)*\cos(t-2)u(t-2) \xleftrightarrow{\mathcal{L}} \frac{e^{-2s}\cdot s}{(s+3)(s^2+1)}$$

Q.19 A signal x(t) has Laplace transform : $X(s)=\frac{s+2}{s^2+4s+5}$

Find the Laplace transform of the following signals.

a) $y_1(t)=\frac{d}{dt}(x(t))$ b) $y_2(t)=x(2t)$

☞ [SPPU : May-19, Marks 6]

Ans. : Given : $X(s)=\frac{s+2}{s^2+4s+5}$

a) $\mathcal{L}\{y_1(t)\}=\mathcal{L}\left[\frac{d}{dt}x(t)\right]=sX(s)=\frac{s(s+2)}{s^2+4s+5}$

b) $\mathcal{L}\{y_2(t)\}=\mathcal{L}\{x(2t)\}=\frac{1}{|a|}X\left(\frac{s}{a}\right)=\frac{1}{2}\left[\frac{\left(\frac{1}{2}\right)+2}{\left(\frac{s}{2}\right)^2+4\left(\frac{s}{2}\right)+5}\right]$

$$Y_2(s)=\frac{s+4}{s^2+8s+20}$$

Q.20 If $X(x)=\frac{2}{(s+3)}$

Find Laplace transform of : 1) $\frac{d}{dt}x(t)$ 2) $t\,x(t)$

☞ [SPPU : Dec.-19, Marks 6]

Ans. : 1) $X(s)=\frac{2}{(s+3)}$

$$\mathcal{L}\left[\frac{d}{dt}x(t)\right]=sX(s)$$

$$\mathcal{L}\left[\frac{d}{dt}x(t)\right]=\frac{2s}{s+3}$$

2) $\mathcal{L}\{t\,x(t)\}=\frac{-dX(s)}{ds}$

$$\therefore\quad \mathcal{L}\{t\,x(t)\}=-\frac{d}{ds}\left[\frac{2}{s+3}\right]=\left[\frac{-2}{(s+3)^2}\right]$$

$$\mathcal{L}\{t\,x(t)\}=\frac{2}{(s+3)^2}$$

5.2 : Unilateral Laplace Transform

Important Points to Remember

- Unilateral Laplace transform is given as,

$$X(s)=\int_{0^-}^{\infty}x(t)e^{-st}\,dt$$

- The unilateral Laplace transform of first three derivatives of x (t) is given as,

$$\frac{dx(t)}{dt}\overset{\mathcal{L}}{\longleftrightarrow}sX(s)-x(0-)$$

$$\frac{d^2x(t)}{dt^2}\overset{\mathcal{L}}{\longleftrightarrow}s^2X(s)-\frac{d}{dt}x(t)\Big|_{t=0-}-sx(0-)$$

$$\frac{d^3x(t)}{dt^3}\longleftrightarrow s^3X(s)-\frac{d^2}{dt^2}x(t)\Big|_{t=0-}\frac{d^2}{dt^2}x(t)\Big|_{t=0-}-s^2x(0-)$$

Q.21 State and prove initial value theorem of Laplace transform.

☞ [SPPU : Dec.-04,22, Marks 3, Dec.-09, Marks 6]

Ans. : If $x(t)\overset{\mathcal{L}}{\longleftrightarrow}X(s)$, then initial value of $x(t)$ is given as,

$$x(0+)=\lim_{t\to 0+}x(t)=\lim_{s\to\infty}[sX(s)] \quad \text{... (Q.21.1)}$$

provided that the first derivative of $x(t)$ should be Laplace transformable.

Proof : From differentiation property of Laplace transform we know that,

$$\mathcal{L}\left[\frac{d}{dt}x(t)\right]=sX(s)-x(0-)$$

Let us take limit of the above equation as $s\to\infty$, i.e.,

$$\lim_{s\to\infty}\mathcal{L}\left[\frac{d}{dt}x(t)\right]=\lim_{s\to\infty}\{sX(s)-x(0-)\} \quad \text{... (Q.21.2)}$$

Consider LHS of above equation i.e.,

$$\lim_{s\to\infty} \mathcal{L}\left[\frac{d}{dt}x(t)\right] = \lim_{s\to\infty}\int_{0-}^{\infty}\frac{d}{dt}x(t)\,e^{-st}\,dt = 0, \text{ since } \lim_{s\to\infty} e^{-st}\,dt = 0$$

Therefore equation (Q.21.2) becomes,

$$0 = \lim_{s\to\infty}\{sX(s)-x(0-)\} = \lim_{s\to\infty}[sX(s)]-x(0-)$$

$$\therefore \quad x(0-) = \lim_{s\to\infty}[sX(s)]$$

$x(0-)$ indicates the value of $x(t)$ just before $t=0$ and $x(0+)$ indicates value of $x(t)$ just after $t=0$. If the function $x(t)$ is continuous at $t=0$, then its value just before and after $t=0$ will be the same. i.e.,

$x(0+)=x(0-)$ for $x(t)$ continuous at $t=0$.

$$\therefore \quad x(0+) = \lim_{s\to\infty}[sX(s)]$$

This equation is used to determine the initial value of $x(t)$ and its derivative.

Q.22 State and prove final value theorem of Laplace transforms.

[SPPU : Dec.-04, Marks 6, Dec.-22, Marks 3]

Ans. : If $x(t)\xleftrightarrow{\mathcal{L}} X(s)$, then final value of $x(t)$ is given as,

$$\boxed{\lim_{t\to\infty} x(t) = \lim_{s\to 0}[sX(s)]} \quad \text{....(Q.22.1)}$$

Proof : From differentiation property we know that,

$$\mathcal{L}\left[\frac{d}{dt}x(t)\right] = sX(s)-x(0-)$$

Let us take limit of the above equation as $s\to 0$, i.e.,

$$\lim_{s\to 0}\mathcal{L}\left[\frac{d}{dt}x(t)\right] = \lim_{s\to 0}\{sX(s)-x(0-)\} = \lim_{s\to 0}\{sX(s)\}-x(0-) \quad \text{... (Q.22.2)}$$

Consider LHS of above equation,

$$\lim_{s\to 0}\mathcal{L}\left[\frac{d}{dt}x(t)\right] = \lim_{s\to 0}\int_{0-}^{\infty}\frac{d}{dt}x(t)\,e^{-st}\,dt = \int_{0-}^{\infty}\frac{d}{dt}x(t)\,dt,$$

since $\lim_{s\to 0} e^{-st} = 1$.

$$= [x(t)]_{0-}^{\infty} = \lim_{t\to\infty}[x(t)-x(0-)] \quad \text{... (Q.22.3)}$$

Hence equation (Q.22.2) can be written as,

$$\lim_{t\to\infty}[x(t)-x(0-)] = \lim_{s\to 0}[sX(s)]-x(0-)$$

$$\therefore \quad \lim_{t\to\infty} x(t) = \lim_{s\to 0}[sX(s)]$$

Q.23 Determine initial value and final values of signal x(t) whose unilateral Laplace transform : X(s) = $\frac{7s+10}{s(s+10)}$.

[SPPU : May-08, Marks 4, Dec.-16, Marks 7, May-18, Marks 3]

Ans. : Initial value is given as,

$$x(0-) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty}\frac{7s+10}{s+10} = \lim_{s\to\infty}\frac{7+\frac{10}{s}}{1+\frac{10}{s}} = 7$$

Final value is given as,

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) = \lim_{s\to 0}\frac{7s+10}{s+10} = 1$$

Q.24 Find the initial value and final value of a signal :
$X(s) = (s+10)(s^2+2s+2)$

[SPPU : May-17, Marks 6]

Ans. : Initial value

$$x(0+) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} s\cdot\frac{s+10}{s^2+2s+2}$$

$$= \lim_{s\to\infty}\frac{s^2+10s}{s^2+2s+2} = \lim_{s\to\infty}\frac{1+\frac{10}{s}}{1+\frac{2}{s}+\frac{2}{s^2}} = 1$$

Final value,

$$x(\infty) = \lim_{s\to\infty} s\cdot X(s) = \lim_{s\to 0} s\cdot\frac{s+10}{s^2+2s+2} = 0$$

Q.25 Find the initial value and final value of the signal x(t) its Laplace transform

$X(s) = \frac{2s+3}{s^2+5s-7}$ [SPPU : May-14, Marks 6, Dec.-18, May-19, Marks 4]

Ans. : $x(0+) = \lim_{s \to \infty} sX(s) = \lim_{s \to \infty} s \cdot \frac{2s+3}{s^2+5s-7}$

$$= \lim_{s \to \infty} \frac{2+\frac{3}{s}}{1+\frac{5}{s}-\frac{7}{s^2}} = 2$$

$$x(\infty) = \lim_{s \to 0} sX(s) = \lim_{s \to 0} s \cdot \frac{2s+3}{s^2+5s-7} = 0$$

Q.26 Determine the initial and final values of the signal having Laplace transform : $X(s) = \frac{2s+3}{s^3+2s^2+5s}$.

[SPPU : Dec.-15, Marks 6, May-18, Marks 3]

Ans. : i) Initial value :

$$\therefore \quad x(0+) = \lim_{s \to \infty} sX(s) = \lim_{s \to \infty} \frac{s(2s+3)}{s^3+2s^2+5s}$$

$$= \lim_{s \to \infty} \frac{2s+3}{s^2+2s+5} = \lim_{s \to \infty} \frac{2+\frac{3}{5}}{s\left(1+\frac{2}{s}+\frac{5}{s^2}\right)} = 0$$

ii) Final value :

$$x(\infty) = \lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s) = \lim_{s \to 0} \frac{s(2s+3)}{s^3+2s^2+5s}$$

$$= \lim_{s \to 0} \frac{2s+3}{s^2+2s+5} = \frac{0+3}{0+0+5} = \frac{3}{5}$$

Q.27 Find the initial and final values for the following function :

$X(s) = \frac{s+5}{s^2+3s+2}$. [SPPU : Dec.-17, June-22, Marks 6]

Ans. : Initial value,

$$x(0+) = \lim_{s \to \infty} sX(s) = \lim_{s \to \infty} s \cdot \frac{s+5}{s^2+3s+2}$$

$$= \lim_{s \to \infty} \frac{s^2+5s}{s^2+3s+2} = \lim_{s \to \infty} \frac{1+\frac{5}{s}}{1+\frac{3}{s}+\frac{2}{s^2}} = 1$$

Final value, $x(\infty) = \lim_{s \to 0} s \cdot X(s) = \lim_{s \to 0} s \cdot \frac{s^2+5}{s^2+3s+2} = 0$

Q.28 Find the unilateral Laplace transform of $x(t) = \cos(\omega_0 t)$.

[SPPU : Dec.-19, Marks 6]

Ans. : $x(t) = \cos(\omega_0 t)$

$$\cos(\omega_0 t) = \left[\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}\right]$$

$$X(s) = \int_0^\infty x(t)e^{-st}\,dt = \frac{1}{2}\left[\int_0^\infty e^{-j\omega_0 t} e^{st}\,dt + \int_0^\infty e^{j\omega_0 t} e^{-st}\,dt\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-j\omega_0}+\frac{1}{s+j\omega_0}\right] = \frac{1}{2}\left[\frac{2s}{s^2+\omega_0^2}\right]$$

$$\therefore \mathcal{L}\,[\cos \omega_0 t] = \frac{s}{s^2+\omega_0^2}$$

5.3 : Inverse Laplace Transform

Important Points to Remember

- The inverse Laplace transform of above equation can be obtained using following standard Laplace transform pairs.

Here we can use following Laplace transform pairs :

$$e^{at}\,u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s-a} \qquad \text{ROC} : Re(s) > a \quad \dots (5.1)$$

$$-e^{at}\,u(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{s-a} \qquad \text{ROC} : Re(s) < a \quad \dots (5.2)$$

- In partial fraction expansion, If there is multiple root of order 'n',

$$X(s) = \frac{N(s)}{(s-s_0)^n}$$

Here there is multiple root of degree 'n' at $s = s_0$. The partial fraction expansion of $X(s)$ will be,

$$X(s) = \frac{k_0}{(s-s_0)^n} + \frac{k_1}{(s-s_0)^{n-1}} + \frac{k_2}{(s-s_0)^{n-2}} + \dots + \frac{k_{n-1}}{(s-s_0)}$$

The values of $k_0, k_1, k_2, \dots$ can be obtained as follows :

$$k_0 = (s-s_0)^n X(s)|_{s=s_0}$$

$$k_1 = \left[\frac{d}{ds}(s-s_0)^n X(s)\right]_{s=s_0}$$

$$k_2 = \frac{1}{2}\left[\frac{d^2}{ds^2}(s-s_0)^n X(s)\right]_{s=s_0}$$

This can be generalized as follows :

$$k_j = \frac{1}{j!}\left[\frac{d^j}{ds^j}(s-s_0)^n\right]_{s=s_0} \quad \dots (5.3)$$

Q.29 Find the inverse Laplace transform of :
$X(s) = -5s-7/(s+1)(s-1)(s+2)$. [SPPU : May-17, Marks 7]

Ans. :
$$X(s) = \frac{A_1}{s+1} + \frac{A_2}{s-1} + \frac{A_3}{s+2}$$
$$= \frac{1}{s+1} - \frac{2}{s-1} + \frac{1}{s+2}$$

Taking inverse Laplace transform,

$$x(t) = e^{-t}\,u(t) - 2e^{t}\,u(t) + e^{-2t}u(t)$$
$$= [e^{-t} - 2e^{t} + e^{-2t}]\,u(t)$$

Q.30 Determine the inverse Laplace transform of :
$X(s) = \frac{2}{s(s+1)(s+2)}$. [SPPU : Dec.-17, Marks 7, June-22, Marks 6]

Ans. :

$$X(s) = \frac{A_1}{s} + \frac{A_2}{s+1} + \frac{A_3}{s+2} = \frac{1}{s} - \frac{1}{s+1} + \frac{1}{s+2}$$

Taking inverse Laplace transform,

$$x(t) = u(t) - 2e^{-t}u(t) + e^{-2t}\,u(t) = \left(1 - 2e^{-t} + e^{-2t}\right)u(t)$$
$$= (1-e^{-t})^2\,u(t)$$

Q.31 Obtain bilateral inverse Laplace transform of the function,
$X(s) = \frac{3s+7}{(s^2-2s-3)}$ **For ROCs of i) $Re(s) > 3$ ii) $Re(s) < -1$**
iii) $-1 < Re(s) < 3$ [SPPU : Dec.-16, May-18, Marks 7]

Ans. : The given function is,

$$X(s) = \frac{3s+7}{s^2-2s-3} = \frac{3s+7}{(s-3)(s+1)}$$

Expanding this function in partial fractions,

$$X(s) = \frac{k_0}{s-3} + \frac{k_1}{s+1} \quad \dots (Q.31.1)$$

Here, $$k_0 = (s-3)\,X(s)\big|_{s=3} = \left.\frac{3s+7}{s+1}\right|_{s=3} = 4$$

and $$k_1 = (s+1)\,X(s)\big|_{s=-1} = \left.\frac{3s+7}{s-3}\right|_{s=-1} = -1$$

$\therefore$ $$X(s) = \frac{4}{s-3} - \frac{1}{s+1} \quad \dots (Q.31.2)$$

i) To obtain inverse Laplace transform for $Re(s) > 3$:

The ROC of $Re(s) > 3$ is right sided as shown in Fig. Q.31.1. Hence the signals will be right sided. Taking the inverse Laplace transform of equation (Q.31.2),

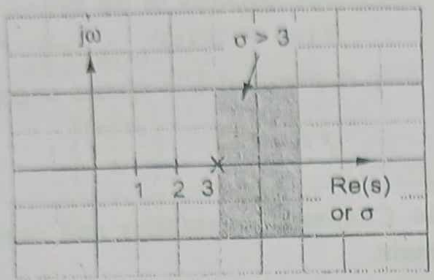


Fig. Q.31.1 ROC of $Re(s) > 3$

$$x(t) = \mathcal{L}^{-1}\left\{\frac{4}{s-3} - \frac{1}{s+1}\right\}$$

$$= 4e^{3t}u(t) - e^{-t}u(t)$$

ii) To obtain inverse Laplace transform for $Re(s) < -1$:

The ROC of $Re(s) < -1$ is left sided. Hence the signals will be left sided. Taking inverse Laplace transform of equation (Q.31.2),

$$x(t) = \mathcal{L}^{-1} X(s) = \mathcal{L}^{-1}\left\{\frac{4}{s-3} - \frac{1}{s+1}\right\} = 4\,\mathcal{L}^{-1}\frac{1}{(s-3)} - \mathcal{L}^{-1}\frac{1}{(s+1)} \quad \text{... (Q.31.3)}$$

Here use, $-e^{at}u(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{s-a}$, ROC : $Re(s) < a$

$$\therefore \quad x(t) = 4\left[-e^{3t}u(-t)\right] - \left[-e^{-t}u(-t)\right]$$

$$= \left(e^{-t} - 4e^{3t}\right)u(-t)$$

iii) To obtain inverse Laplace transform for $-1 < Re(s) < 3$:

The ROC of $-1 < Re(s) < 3$ is shown in Fig. Q.31.2. This ROC can also be expressed as, $Re(s) < 3$ and $Re(s) > -1$. And,

$$X(s) = \frac{4}{s-3} - \frac{1}{s+1} \quad \text{... (Q.31.4)}$$

This function has two poles : Poles at s = 3 and s = –1

The pole at $s = -1$ is to left side of the ROC. Hence the term corresponding to this ROC will be right sided. i.e.,

$$\mathcal{L}^{-1}\frac{1}{s+1} = e^{-t}u(t) \text{ for ROC : } Re(s) > -1$$

Similarly, the pole at $s = 3$ is to right side of the ROC. Hence the term corresponding to this ROC will be left sided. i.e.,

$$\mathcal{L}^{-1}\frac{1}{s-3} = -e^{3t}u(-t) \text{ for ROC : } Re(s) < 3$$

Hence Laplace inverse of equation (Q.31.2) becomes,

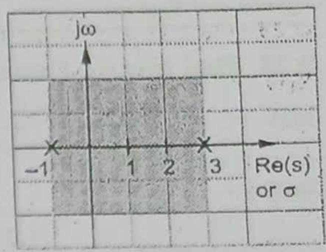


Fig. Q.31.2 ROC of $-1 < Re(s) < 3$

$$x(t) = 4\left[-e^{3t}u(-t)\right] - e^{-t}u(t)$$

$$= -4e^{3t}u(-t) - e^{-t}u(t)$$

Q.32 Find the inverse Laplace transform of :

$$\mathbf{X(s)} = \frac{2}{(s+4)(s-1)}.$$

If the region of convergence is :

i) $-4 \le Re(s) < 1$ ii) $Re(s) > 1$ iii) $Re(s) < -4$.

[SPPU : Dec.-14,18,22, May-19, Marks 6]

Ans. : i) $x(t)$ for ROC $-4 \le Re(s) < 1$

$$X(s) = \frac{2/5}{s-1} - \frac{2/5}{s+4}$$

Here poles are at $s = 1$ and $s = -4$. Note that ROC lies to left of $s = 1$ and right of $s = -4$. Hence inverse Laplace transform will be,

$$x(t) = -\frac{2}{5}e^{t}u(-t) - \frac{2}{5}e^{-4t}u(t)$$

ii) $x(t)$ for ROC $Re(s) > 1$

ROC lies at right side of both poles. Hence inverse Laplace transform will be,

$$x(t) = \frac{2}{5}e^{t}u(t) - \frac{2}{5}e^{-4t}u(t)$$

iii) $x(t)$ for ROC $Re(s) < -4$

ROC lies at left side of both poles. Hence,

$$x(t) = -\frac{2}{5}e^{t}u(-t) + \frac{2}{5}e^{-4t}u(-t)$$

Q.33 Find inverse Laplace Transform of

$$X(s) = \frac{3s^2 + 8s + 6}{(s+2)(s^2+2s+1)}$$

☞ **[SPPU : Dec.-19, Marks 6]**

Ans. :

$$X(s) = \frac{3s^2+8s+6}{(s+2)(s^2+2s+1)} = \frac{3s^2+8s+6}{(s+2)(s+1)^2}$$

$$= \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

$$A = (s+2)\frac{3s^2+8s+6}{(s+2)(s+1)^2}\bigg|_{s=-2}$$

$$A = 2$$

$$B = \frac{1}{1!}\frac{d}{ds}\left[(s+1)^2 \frac{3s^2+8s+6}{(s+2)(s+1)^2}\right]$$

$$B = 1$$

$$C = (s+1)^2 \frac{3s^2+8s+6}{(s+1)^2 (s+2)}\bigg|_{s=-1}$$

$$C = 1$$

$$X(s) = \frac{2}{s+2} + \frac{1}{s+1} + \frac{1}{(s+1)^2}$$

$$x(t) = 2e^{-2t} u(t) + e^{-t} u(t) + t e^{-t} u(t)$$

5.4 : Application of Laplace Transforms to LTI System Analysis

Important Points to Remember

- The system function is the ratio of Laplace transforms of output to input i.e,

$$H(s) = \frac{Y(s)}{X(s)}$$

- Frequency response of the system is given as,

$$H(\omega) = H(s)\big|_{s=j\omega}$$

- Step response of LTI system is given as,

$$y(t) = \mathcal{L}^{-1}\left\{\frac{H(s)}{s}\right\}$$

Q.34 Comment on stability and causality of LTI system in frequency domain.

Ans. : Causality : For right sided ROC, time domain signals are right sided. Or, when all the poles are to the left side of ROC, then time domain signals are right sided. A right sided impulse response represents causal system. Hence we can state,

An LTI system is said to be causal if all the poles of it's system function lie on left side of the ROC.

Stability : We have seen earlier that, the system is said to be stable, if its impulse response is absolutely integrable. i.e.,

$$\sum_{t=-\infty}^{\infty} |h(t)| < \infty$$

If above equation is satisfied, then Fourier transform of $h(t)$ exists. We know that Fourier transform is obtained from Laplace transform for $Re(s) = 0$. Hence ROC of $H(s)$ must include $Re(s) = 0$, i.e. $j\omega$ axis. Thus,

An LTI system is said to be stable if and only if the ROC of its system function includes $Re(s) = 0$, i.e. $j\omega$ axis of s-plane.

Causality states that all the poles must lie on left side of ROC. Hence the system will be causal and stable simultaneously if and only if all the poles of $H(s)$ lie on left side of $j\omega$ axis or left half of s-plane.

Q.35 The differential equation of the system is given by : $dy(t)/dt + 2y(t) = x(t)$. Determine the output of system for $x(t) = e^{-3t} u(t)$. Assume zero initial condition.

☞ **[SPPU : May-17, Dec.-22, Marks 6]**

Ans. : Taking Laplace transform of given differential equation and input we get,

$$sY(s)+2Y(s) = X(s) \quad \text{here } X(s) = \frac{1}{s+3}$$

$$\therefore \quad (s+2)Y(s) = \frac{1}{s+3}$$

$$\therefore \quad Y(s) = \frac{1}{(s+2)(s+3)} = \frac{1}{s+2} - \frac{1}{s+3}$$

Taking inverse Laplace transform

$$y(t) = (e^{-2t} - e^{-3t})u(t)$$

Q.36 Find transfer function and impulse response of the causal system described by the differential equation :

$$\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 6y(t) = 2\frac{d}{dt}x(t) - 3x(t)$$

[SPPU : May-18, Marks 6]

Ans. : Taking Laplace transform of given differential equation,

$$s^2Y(s) + 5\cdot sY(s) + 6Y(s) = 2\cdot sX(s) - 3X(s)$$

$$\therefore \quad Y(s)[s^2+5s+6] = X(s)[2s-3]$$

$$H(s) = \frac{2s-3}{s^2+5s+6} \quad \text{(Transfer function)}$$

$$H(s) = \frac{9}{s+3} - \frac{7}{s+2}$$

Taking inverse Laplace transform,

$$h(t) = [9e^{-3t} - 7e^{-2t}]u(t)$$

FORMULAE AT A GLANCE

i) Laplace transform :

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt \quad \text{... (5.4)}$$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\, e^{st}\, ds \quad \text{... (5.5)}$$

ii) Unilateral Laplace transform

$$X(s) = \int_{0-}^{\infty} x(t)\, e^{-st}\, dt \quad \text{... (5.6)}$$

$$x(0+) = \lim_{t\to 0+} x(t) = \lim_{s\to\infty} [sX(s)] \quad \text{... (5.7)}$$

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} [sX(s)] \quad \text{... (5.8)}$$

END...

Unit VI

6 Probability and Random Variables

6.1 : Probability

Important Points to Remember

- An experiment is the process which is conducted to get some results.
- A set of all possible outcomes of an experiment is called sample space of that experiment.
- The expected subset of the sample space or happening is called an event.
- Probability of event 'A' is defined is the ratio of number of possible favourable outcomes to total number of outcomes.
- Combination of 'n' taken 'r' at a time, $^nC_r = \dfrac{n!}{(n-r)!r!}$ ... (6.1)
- Permutations of 'n' taken 'r' at a time, $^nP_r = \dfrac{n!}{(n-r)!}$ ... (6.2)
- The joint probability of events A and B occuring simultaneously is denoted by $P(AB)$. It is given as,

$$P(AB) = \lim_{N \to \infty} \frac{N_{AB}}{N}$$

- Conditional probability $P(B/A) = \dfrac{P(AB)}{P(A)}$ or $P(A/B) = \dfrac{P(AB)}{P(B)}$
- If occurrance of event 'B' doesnot depend upon occurance of event 'A', then these two events are called statically independent.

$$P(B/A) = P(B) \text{ and } P(A/B) = P(A)$$

$$\therefore \quad P(AB) = P(A) - P(B).$$

Q.1 Define and explain axioms of probability.

[SPPU : Dec.-04, May-05, Dec.-16, Marks 4; Dec.-18, Marks 3, Dec.-22, Marks 5]

Ans. : **(1)** If the event contains all outcomes, it is certain event i.e.,
$P(A) = P(S) = 1$

(2) Probability of any event lies between '0' and '1'.

$0 \le P(A) \le 1$

(3) If 'A' and 'B' are mutually exclusive events,

$P(A+B) = P(A) + P(B))$

(4) Property 1 : $P(\overline{A}) = 1 - P(A)$ ...(Q.1.1)

Here $\overline{A}$ denotes the complement of event A.

(5) Property 2 : If $A_1, A_2, \ldots, A_M$ are mutually exclusive events,

Then $P(A_1) + P(A_2) + \ldots P(A_M) = 1$ ...(Q.1.2)

(6) Property 3 : If events A and B are not mutually exclusive events then the probability of the union of A or B is given as,

$$P(A+B) = P(A) + P(B) - P(AB) \quad \text{...(Q.1.3)}$$

Here $P(AB)$ is called the probability of events A and B both occurring simultaneously. Such event is called joint event of A and B and the probability $P(AB)$ is called joint probability it is defined as,

$$P(AB) = \lim_{N \to \infty} \frac{N_{AB}}{N} \quad \text{...(Q.1.4)}$$

If events A and B are mutually exclusive, then the joint probability $P(AB) = 0$.

Q.2 A box contains 3 white, 4 red and 5 black balls. A ball is drawn at random find the probability that it is 1) Red 2) Not black 3) Black or white.

[SPPU : May-19, June-22, Marks 3]

Ans. : There are total 12 balls. Hence one ball can be drawn from 12 balls in, $^{12}C_1$ ways. i.e.,

$$N = {}^{12}C_1 = 12$$

i) P(red)

Out of 4 red balls one ball can be drawn in $^4C_1$ ways. i.e.,

$$N(\text{red}) = {}^4C_1 = 4$$

$$\therefore \quad P(\text{red}) = \frac{N(\text{red})}{N} = \frac{4}{12} = \frac{1}{3}$$

ii) P (not black) :

Then probability that ball will not be black is same as probability that it will be white or red. Hence,

$$N(\text{red}) = 4$$

$$N(\text{white}) = {}^{3}C_1 = 3$$

$$\therefore \quad P(\text{not black}) = P(\text{red}) + P(\text{white})$$

$$= \frac{N(\text{red})}{N} + \frac{N(\text{white})}{N} = \frac{4}{12} + \frac{3}{12} = \frac{7}{12}$$

iii) P(black or white)

Here $N(\text{black}) = {}^{5}C_1 = 5$

and $N(\text{white}) = {}^{3}C_1 = 3$

$$P(\text{black or white}) = P(\text{black}) + P(\text{white})$$

$$= \frac{N(\text{black})}{N} + \frac{N(\text{white})}{N}$$

$$= \frac{5}{12} + \frac{3}{12} = \frac{8}{12} = \frac{2}{3}$$

Q.3 In a pack of cards, 2 cards are drawn simultaneously. What is the probability of getting a queen, jack combination ?

☞ [SPPU : May-17, Marks 6, Dec.-22, Marks 4]

Ans. : Number of ways of drawing 52 cards in a pack,

$$N = {}^{52}C_2 = \frac{52!}{2!(52-2)!} = 1326 \text{ ways}$$

Number of ways of drawing a queen from four queens,

$$N_q = {}^{4}C_1 = \frac{4!}{1!3!} = 4 \text{ ways}$$

Number of ways of drawing a jack from four jacks,

$$N_j = {}^{4}C_1 = \frac{4!}{1!3!} = 4 \text{ ways}$$

Probability that two cards drawn will be queen-jack combination is,

$$P = \frac{N_q \cdot N_j}{N} = \frac{4 \times 4}{1326} = 12.06 \times 10^{-3}$$

Q.4 A perfect die is thrown. Find the probability that : i) You get even number ii) You get a perfect square.

☞ [SPPU : Dec.-16, Marks 7, Dec.-22, Marks 6]

Ans. : Sample space for throwing a perfect die will be,

$$S = \{1, 2, 3, 4, 5, 6\} \text{ Hence } N = 6 \text{ samples}$$

i) Getting even number, $A = \{2, 4, 6\}$ Hence $N_A = 3$

$$P(A) = \frac{N_A}{N} = \frac{3}{6} = 0.5$$

ii) Getting a perfect square, B = {1, 4}, Hence N = 2

$$P(B) = \frac{N_B}{N} = \frac{2}{6} = 0.333$$

Q.5 A box contains 10 white, 15 red and 15 black balls. A ball is drawn at random find the probability that it is : 1) Red 2) Not black 3) Black or white.

☞ [SPPU : Dec.-19, Marks 3]

Ans. :

No. of white balls = 10

No. of red balls = 15

No. of black balls = 15

Total number of balls = 40

$$P(\text{Red}) = \frac{15}{40} = \frac{3}{8}$$

$$P(\text{Black or white}) = \frac{25}{40} = \frac{3}{8}$$

$$P(\text{Not black}) = \frac{25}{40} = \frac{5}{8}$$

6.2 : Probability Models

Important Points to Remember

- Binomial PDF and CDF are given as,

$$f_X(x) = \sum_{k=0}^{n} {}^nC_k\, p^k (1-p)^{n-k}\, \delta(x-k)$$

$$F_X(x) = \sum_{k=0}^{n} {}^nC_k\, p^k (1-p)^{n-k}$$

- Thermal noise generated in electronic components have Gaussian or normal distribution. The random errors in the experimental mesurements cause the measured values to have Gaussian distribution about the true value.

Q.6 Explain uniform distribution model with respect to its density and distribution function.

[SPPU : May-14, Marks 8, May-16, Marks 2, Dec.-07, May-19, Marks 4, Dec.-09, Marks 8, May-06, Dec.-11, Marks 3]

Ans. : Uniform distribution is used to represent quantization noise in PCM and similar other phenomenon.

The PDF for a uniform distribution is given as,

$$\textbf{Uniform PDF : } f_X(x) = \begin{cases} 0 & for\ x < m-\frac{A}{2}\ and\ x > m+\frac{A}{2} \\ \frac{1}{A} & for\ \left(m-\frac{A}{2}\right) \le x \le \left(m+\frac{A}{2}\right) \end{cases} \quad \text{... (Q.6.1)}$$

- And CDF is given as,

$$f_X(x) = \begin{cases} 0 & \text{for} \quad x < m-\frac{A}{2} \\ \frac{1}{A}\left\{x-m+\frac{A}{2}\right\} & \text{for} \quad m-\frac{A}{2} \le x \le m+\frac{A}{2} \\ 1 & \text{for} \quad x \ge m+\frac{A}{2} \end{cases}$$

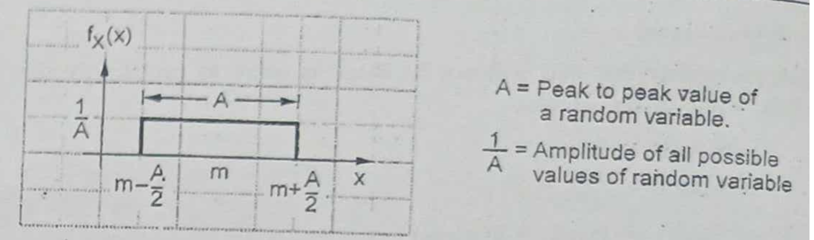


Fig. Q.6.1 PDF of uniformly distributed random variable. The peak to peak value is 'A' and amplitude is uniform (i.e. A)

- The value of PDF, $f_X(x)$ is same for all possible values of a random variable. therefore this distribution is called *Uniform Distribution.*
- The mean and variance of random variable 'X' having uniform distribution in interval [a, b] are $m_x = \frac{a+b}{2}$ and $\sigma_x^2 = \frac{(a-b)^2}{12}$

Q.7 Explain the Gaussian or normal probability model with respect to its density and distribution functions.

[SPPU : May-14,19, Marks 8, May-16, Marks 2, Dec.-07, 08, 11, May-12 Marks 4, Dec.-05, 13, Marks 6, Dec.-10, Marks 5, May-15, Dec.-19, Marks 4]

Ans. : Gaussian distribution is also called *Normal Distribution*. It is defined for continuous random variables. The *PDF* for a Gaussian random variable is given as,

$$\textbf{Gaussian PDF : } f_X(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-m)^2/2\sigma^2} \quad \text{... (Q.7.1)}$$

Here 'm' is mean and σ^2 is variance.

Fig. Q.7.1 shows the sketch of Gaussian pdf.

Properties of Gaussian *PDF*

Property 1 : The peak value occurs at $x = m$ (i.e. mean value). i.e.,

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \quad \text{at} \quad x = m \quad \text{i.e. mean value} \quad \text{... (Q.7.2)}$$

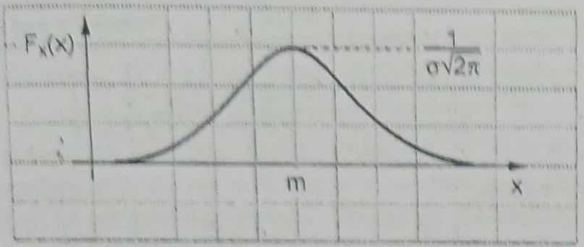


Fig. Q.7.1 Plot of Gaussian *PDF*

Property 2 : The plot of Gaussian *PDF* has even symmetry around mean value i.e.,

$$f_X(m-\sigma) = f_X(m+\sigma) \quad \text{... (Q.7.3)}$$

Property 3 : The area under the *PDF* curve is 1/2 for all values of x below mean value and 1/2 for all values of x above mean value. i.e.,

$$P(X \leq m) = P(X > m) = \frac{1}{2} \quad \text{... (Q.7.4)}$$

Property 4 : As $\sigma \to 0$ the Gaussian function approaches to δ (i.e. impulse) function located at $x = m$. This is because the area under the *PDF* curve is always unity. And the area of impulse function is also unity.

Q.8 A student arrives late for a class 40 % of the time. Class meets five time each week. Find : i) Probability if students being late for at least three classes in a given week. ii) Probability of students will not be late at all during a given week. [SPPU : May-15, Marks 4]

Ans. : Here let the probability that the student will be let to the class is

$$p = \frac{40}{100} = 0.4.$$

$1-p$ is probability that student do not come late. It will be

$1-p = 1-0.4 = 0.6.$

n is number of times class meets in a week. Hence n = 5.

k is number of times student is late. Here, $k = 3$.

i) Probability that student gets late for at least 3 classes in a week

Probability that student gets late for at least three classes in a week can be expressed as, $P(X \geq 3) = P(X=3)+P(X=4)+P(X=5)$

We know that , $P(X=k) = {}^nC_k\, p^k (1-p)^{n-k}$

$$\therefore P(X \geq 3) = {}^5C_3 (0.4)^3 (0.6)^2 + {}^5C_4 (0.4)^4 (0.6)^1 + {}^5C_5 (0.4)^5 (0.6)^0$$

$$= \frac{5!}{(5-3)!3!}(0.4)^3 (0.6)^2 + \frac{5!}{(5-4)!4!}(0.4)^4 (0.6) + \frac{5!}{(5-5)!5!}(0.4)^5 \cdot 1$$

$= 0.317$

ii) Probability that a student will not be late at all

Here k = 0; the number of times student is late.

$$\therefore P(X=0) = {}^5C_0\, p^0 (1-p)^5 = {}^5C_0 (0.4)^0 (0.6)^5$$

$$= \frac{5!}{(5-0)!0!} \times 1 \times 0.6^5$$

$$= 0.0778$$

Q.9 Show that the mean and variance of a random variable 'X' having a uniform distribution in the interval [a, b] are,

$$m_x = \frac{a+b}{2} \quad \text{and} \quad \sigma_x^2 = \frac{(a-b)^2}{12}$$

[SPPU : Dec.-08, May-12, Marks 6, Dec.-09, Marks 8, Dec.-14, Marks 4, May-16, Marks 4]

Ans. : Fig. Q.9.1 shows the sketch of uniform distribution in the interval [a, b].

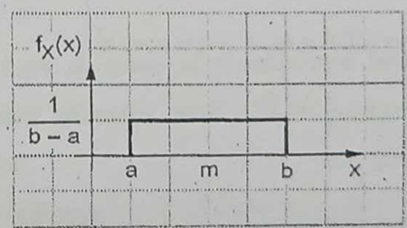


Fig. Q.9.1 Uniform distribution having interval [a, b]

To find mean value

The mean value of a continuous random variable is given as,

$$m_x = \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_a^b x \cdot \frac{1}{b-a}\, dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{1}{2(b-a)}\left[b^2-a^2\right] = \frac{1}{2(b-a)}\,(b-a)\cdot(b+a)$$

$$= \frac{b+a}{2} \quad or \quad \frac{a+b}{2} \qquad \text{...(Q.9.1)}$$

To find variance

Variance is given as,

$$\sigma_x^2 = \int_{-\infty}^{\infty} (x-m_x)^2\, f_X(x)\, dx$$

$$= \int_a^b (x-m_x)^2 \cdot \frac{1}{b-a}\, dx \quad \text{since } f_X(x) = \frac{1}{b-a}$$

Let $x - m_x = y$ then we have $dx = dy$.

And the limits will be,

when $x = a$, $y = a - m_x$ and when $x = b$, $y = b - m_x$

$$\therefore \quad \sigma_x^2 = \int_{a-m_x}^{b-m_x} y^2 \cdot \frac{1}{b-a}\, dy = \frac{1}{b-a}\left[\frac{y^3}{3}\right]_{a-m_x}^{b-m_x}$$

$$= \frac{1}{3(b-a)}\left[(b-m_x)^3 - (a-m_x)^3\right]$$

Putting the value of $m_x = \frac{a+b}{2}$ from equation (Q.9.1) in above equation we get,

$$\sigma_x^2 = \frac{1}{3(b-a)}\left[\left(b-\frac{a+b}{2}\right)^3 - \left(a-\frac{a+b}{2}\right)^3\right]$$

$$= \frac{1}{3(b-a)}\left[\left(\frac{b-a}{2}\right)^3 - \left(\frac{a-b}{2}\right)^3\right]$$

$$= \frac{1}{-3(a-b)}\left[\left(-\frac{a-b}{2}\right)^3 - \left(\frac{a-b}{2}\right)^3\right]$$

Here we have written,

$$b-a = -(a-b) = \frac{1}{-3(a-b)} \times \frac{-(a-b)^3}{4} = \frac{(a-b)^2}{12} \qquad \text{... (Q.9.2)}$$

6.3 : Random Variables

Important Points to Remember

- **Random variable :** A function which takes on any value from the sampe space and its range is some of real numbers is called a random variable of the experiment.
- **Continuous random variable :** If the random 'X' takes on any value in a whole observation interval, X is called a continuous random variable.
- **Discrete random variables :** The random variable X is a discrete random variable if X can take only finite number of values in any finite observation interval. Thus the discrete random variable has countable number of distinct values.
- CDF, $F_X(x) = \begin{cases} 0 & \text{for } x < x_1 \\ \sum_{i=1}^{n} P(X = x_i) & \text{for } x_1 \le x \le x_n \\ 1 & \text{for } x \ge x_n \end{cases}$

Q.10 State and explain the properties of CDF.

[SPPU : Dec.-15 Marks 6, Dec.-14, June-22, Marks 3, May-14, Dec.-05, Dec.-07, Marks 4]

Ans. : *Definition : The Cumulative Distribution Function (CDF) of a random variable 'X' is the probability that a random variable 'X' takes a value less than or equal to x.*

Thus,

$$CDF : F_X(x) = P(X \le x) \qquad \text{... (Q.10.1)}$$

Property 1 : The *CDF* is bounded between 0 and 1. i.e.,

$$0 \le F_X(x) \le 1 \qquad \text{... (Q.10.2)}$$

Since CDF is the probability distribution function i.e. it is defined as probability of some event $\{X \leq x\}$, its value is always between 0 and 1.

Property 2 : $F_X(-\infty) = 0$ and $F_X(\infty) = 1$. ... (Q.10.3)

Here $x = -\infty$ means no possible event. Hence $P(X \leq -\infty)$; will be always zero. Therefore $F_X(-\infty) = 0$. At $x = \infty$ means $P(X \leq \infty)$ that includes probability of all possible events. Therefore probability of certain event is '1'. Hence $F_X(\infty) = 1$.

Property 3 : $F_X(x_1) \leq F_X(x_2)$ if $x_1 \leq x_2$ ... (Q.10.4)

It states that the CDF F_X (x) is a monotonic non decreasing function of x.

Q.11 State and explain the properties of PDF.

[SPPU : Dec.-10,11,18, Marks 4, May-05, Marks 2, May-14, Marks 8, Dec.-13, May-15,19, June-22 Marks 3, May-16, Dec.-17, Marks 6]

Ans. : Definition : The derivative of Cumulative Distribution Function (CDF) with respect to some dummy variable is called as Probability Density Function (PDF).

$$\text{PDF : } f_X(x) = \frac{d}{dx} F_X(x) \quad \text{... (Q.11.1)}$$

Property 1 : PDF is nonzero for all values of x.

i.e. $f_X(x) \geq 0$ for all x ... (Q.11.2)

Proof : Since Cumulative Distribution Function (*CDF*) increases monotonically, the derivative of *CDF* will always be positive. *PDF* is obtained by taking derivative of CDF. Hence *PDF* will be always positive.

Property 2 : The area under the *PDF* curve is equal to 1. i.e.,

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1 \quad \text{... (Q.11.3)}$$

Proof : $f_X(x) = \frac{d}{dx} F_X(x)$

$$\therefore \int_{-\infty}^{\infty} f_X(x) dx = \int_{-\infty}^{\infty} \left[\frac{d}{dx} F_X(x)\right] dx = [F_X(x)]_{-\infty}^{\infty} = [F_X(\infty) - F_X(-\infty)]$$

$= 1 - 0 = 1$

Property 3 : *CDF* is obtained by integrating *PDF*. i.e.,

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx \quad \text{... (Q.11.4)}$$

Proof : $f_X(x) = \frac{d}{dx} F_X(x)$

$$\therefore \int_{-\infty}^{x} f_X(x)\, dx = \int_{-\infty}^{x} \left[\frac{d}{dx} F_X(x)\right] dx = [F_X(x)]_{-\infty}^{x}$$

$= [F_X(x) - F_X(-\infty)] = F_X(x) - 0 = F_X(x)$

Property 4 : Probability of the event $\{x_1 < X \leq x_2\}$ is simply given by area under the PDF curve in the range $x_1 < X \leq x_2$.

Proof : The probability of $\{X \leq x_2\}$ can be expressed as sum of mutually exclusive events $\{X \leq x_1\}$ and $\{x_1 < X \leq x_2\}$.

i.e., $P(X \leq x_2) = P(X \leq x_1) + P(x_1 < X \leq x_2)$ ... (Q.11.5)

$\therefore \quad F_X(x_2) = F_X(x_1) + P(x_1 < X \leq x_2)$ since $F_X(x) = P(X \leq x)$

$\therefore P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$ ... (Q.11.6)

$$P(x_1 < X \leq x_2) = \int_{-\infty}^{x_2} f_X(x)\, dx - \int_{-\infty}^{x_1} f_X(x)\, dx \quad \text{... By equation (Q.11.4)}$$

$$= \int_{x_1}^{x_2} f_X(x)\, dx \quad \text{... (Q.11.7)}$$

Q.12 : A three digit message is transmitted over a noisy channel having a probability of error as $P(E) = \frac{2}{5}$ per digit. Find out the corresponding CDF. [SPPU : Dec.-16, Marks 7]

Ans. : $P(E) = \frac{2}{5}$ per digit

$\therefore$ Probability of the correct digit is,

$$P(C) = 1 - P(E) = 1 - \frac{2}{5} = \frac{3}{5} \text{ per digit}$$

Let 'C' denote correct digit and 'E' denote error digit. The sample space for 3 digits will be,

$$S = \{CCC, CCE, CEC, CEE, ECC, ECE, EEC, EEE\} \quad \text{... (Q.12.1)}$$

The random variable 'X' represents '4' values as follows :

X = {no error (x_0), one digit in error (x_1), two digits in error (x_2), all digits in error (x_3)} ...(Q.12.2)

$P(X = X_0)$: Probability of no error

$$P(X = X_0) = P(C)\,P(C)\,P(C) \times 1$$

P (C) is probability for one correct digit

$$= \left(\frac{3}{5}\right)\left(\frac{3}{5}\right)\left(\frac{3}{5}\right) = \frac{27}{125}$$

$P(X = X_1)$: Probability of one digit in error

There are total three sample points in the sample space S which have one digit in error. Random variable X maps all these sample points at $x = x_1$.

i.e. $x_1 = \{CCE, CEC, ECC\}$

$$\therefore \; P(X = x_1) = [P(C)P(C)P(E)] \times [P(C)P(E)P(C)] \, [P(E)P(C)P(C)]$$

or $P(X = x_1) = [P(C)\,P(C)\,P(E)]$

× (Number of sample points with one digit in error)

$$= [P(C)\,P(C)\,P(E)] \times 3 = \left(\frac{3}{5}\right)\left(\frac{3}{5}\right)\left(\frac{2}{5}\right) \times 3 = \frac{54}{125}$$

$P(X = X_2)$ = Probability of two digits in error

There are total three sample points which are mapped in $x = x_2$.

i.e. $x_2 = \{CEE, ECE, EEC\}$

$P(X = X_2) = [P(C)P(E)P(E)]$ (Number of sample points with two digits in error)

$$= \left(\frac{3}{5}\right)\left(\frac{2}{5}\right)\left(\frac{2}{5}\right) \times 3 = \frac{36}{125}$$

$P(X = X_3)$: Probability of three digits in error

There is only one sample point having all three digits in error i.e.

$x_3 = \{EEE\}$

$$P(X = X_3) = [P(E)\,P(E)\,P(E)] \times 1 = \left(\frac{2}{5}\right)\left(\frac{2}{5}\right)\left(\frac{2}{5}\right) \times 1 = \frac{8}{125},$$

Calculation of CDF

$F_X(x) = 0$ for $x < x_0$ i.e., No possible events

$$F_X(x_0) = P(X \le x_0) = P(X < x_0) + P(X = x_0)$$

$$= 0 + \frac{27}{125} = \frac{27}{125}$$

$$F_X(x_1) = P(X \le x_1) = P(X < x_0) + P(X = x_0) + P(X = x_1)$$

$$= 0 + \frac{27}{125} + \frac{54}{125} = \frac{81}{125}$$

$$F_X(x_2) = P(X \le x_2)$$

$$= P(X \le x_0) + P(X < x_0) + P(X = x_1) + P(X = x_2)$$

$$= 0 + \frac{27}{125} + \frac{54}{125} + \frac{36}{125} = \frac{117}{125}$$

$$F_X(x_3) = P(X \le x_3)$$

$$= P(X \le x_0) + P(X < x_0) + P(X = x_1) + P(X = x_2) + P(X = x_3)$$

$$= 0 + \frac{27}{125} + \frac{54}{125} + \frac{36}{125} + \frac{8}{125} = \frac{125}{125} = 1$$

Fig. Q.12.1 shows the probabilities of random variable X and corresponding Cumulative Distribution Function (CDF).

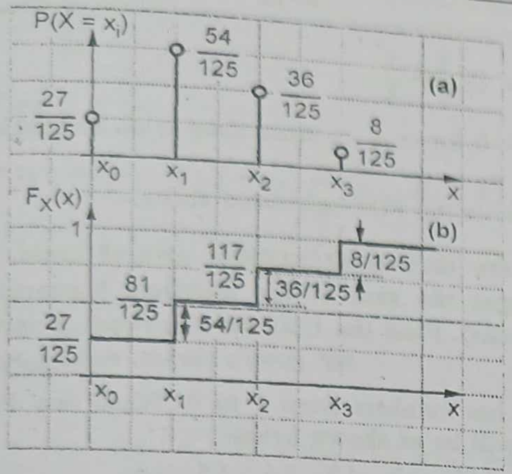


Fig. Q.12.1 : Sketch of CDF

Q.13 A coin is tossed 3 times. Write the sample space which gives all possible outcomes. A random variable X, represents the number of heads on any triple toss. Calculate and draw the CDF and PDF.

[SPPU : Dec.-09, Marks 10, May-11, Marks 8, Dec.-18, Marks 6, Dec.-19,22, Marks 7]

Ans. : The sample space of all possible outcomes will be as follows :

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

The random variable 'X' represents number of heads in any tripple toss. Hence possible values of random variable will be,

$$X = \left\{\underbrace{No\ heads}_{x_0}, \underbrace{One\ head}_{x_1}, \underbrace{Two\ heads}_{x_2}, \underbrace{Three\ heads}_{x_3},\right\}$$

CDF

The CDF is given as,

$$F_X(x) = \begin{cases} 0 & \text{for } x < x_0 \\ \sum_{i=1}^{2} P(X = x_i) & \text{for } x_0 \le x \le x_3 \\ 1 & \text{for } x > x_3 \end{cases}$$

$$= P(X = x_0) + P(X = x_1) + P(X = x_2) + P(x_3)$$

$P(X = x_0)$: Probability of getting no heads : In sample space there is only one outcome that has all tails. i.e.,

$$x_0 = \{T, T, T\}$$

Probability of getting head or tail is $\frac{1}{2}$. Hence,

$$P(X = x_0) = P(T) \cdot P(T) \cdot P(T) \times \text{Number of outcomes for } x_0$$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times 1 = \frac{1}{8}$$

$P(X = x_1)$: Probability of getting one head : In sample space there are three outcomes that has one head i.e.,

$$\therefore \quad x_1 = \{HTT, THT, TTH\}$$

$$\therefore \quad P(X = x_1) = P(H) \cdot P(T) \cdot P(T) \times \text{Number of outcomes for } x_1$$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times 3 = \frac{3}{8}$$

$P(X = x_2)$: Probability of getting two heads : In sample space there are 3 outcomes that has two heads. i.e.,

$$\therefore \quad x_2 = \{HHT, HTH, THH\}$$

$$\therefore \quad P(X = x_2) = P(H) \cdot P(H) \cdot P(T) \times \text{Number of outcomes for } x_2$$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times 3 = \frac{3}{8}$$

$P(X = x_3)$: Probability of getting all heads : In sample space only one outcome has all heads. i.e.,

$$x_3 = \{HHH\}$$

$$P(X = x_3) = P(H) \cdot P(H) \cdot P(H) \times \text{Number of outcomes for } x_3$$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times 1 = \frac{1}{8}$$

Now CDF can be calculated as follows :

$$F_X(x) = 0 \quad \text{for } x < x_0$$

$$F_X(x_0) = P(X \le x_0) = P(X < x_0) + P(X = x_0)$$

$$= 0 + \frac{1}{8} = \frac{1}{8}, \quad \text{since } P(X = x_0) = 0$$

$$F_X(x_1) = P(X \le x_1) = P(X < x_0) + P(X = x_0) + P(X = x_1)$$

$$= 0 + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

Similarly, $F_X(x_2) = P(X \le x_2)$

$$= P(X < x_0) + P(X = x_0) + P(X = x_1) + P(X = x_2)$$

$$= 0 + \frac{1}{8} + \frac{3}{8} + \frac{3}{8} = \frac{7}{8}$$

$$F_X(x_3) = P(X \le x_3)$$

$$= P(X < x_0) + P(X = x_0) + P(X = x_1) + P(X = x_2) + P(X = x_3)$$

$$= 0 + \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$$

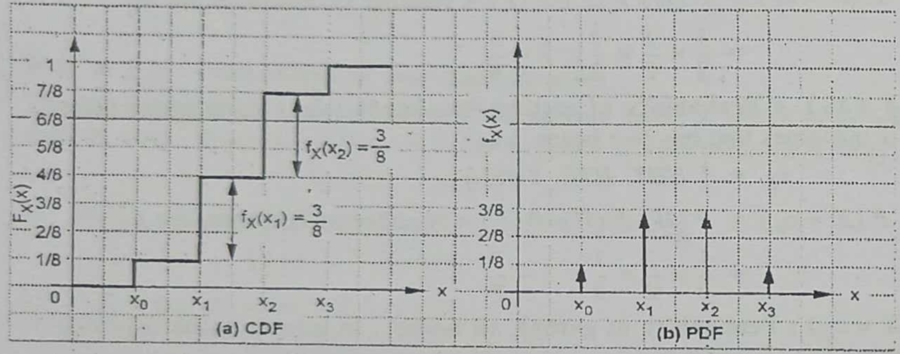


Fig. Q.13.1 : Plots of CDF and PDFs

Fig. Q.13.1 (a) shows the CDF of X.

PDF

$f_X(x) = \frac{d}{dx} F_X(x)$ i.e. difference in amplitudes of CDF plot of Fig. Q.13.1.

$f_X(x) = 0$ for $x < x_0$, since there is no amplitude difference in CDF plot for $x < x_0$

$$f_X(x_0) = \frac{1}{8} - 0 = \frac{1}{8}$$ i.e. difference in CDF amplitudes at x_0

$$f_X(x_1) = \frac{4}{8} - \frac{1}{8} = \frac{3}{8}$$

$$f_X(x_2) = \frac{7}{8} - \frac{4}{8} = \frac{3}{8}$$

$$f_X(x_3) = 1 - \frac{7}{8} = \frac{1}{8}$$

$f_X(x) = 0$ for $x < x_3$ since there is no amplitude difference in CDF plot for $x > x_3$

Fig. Q.13.1 (b) shows the PDF calculated above.

Q.14 Two dice are thrown at random several times. The random variable 'X' assigns the sum of the numbers appearing on dice to each outcome (event). Find the CDF for the random variable.

[SPPU : Dec.-11, Marks 8, May-19, Marks 6]

Ans. : Each dice has numbers from 1 to 6. When two dice are thrown, the sample space will be as shown below :

$$S = \begin{Bmatrix} 1,1, & 1,2 & 1,3 & 1,4 & 1,5 & 1,6 \\ 2,1 & 2,2 & 2,3 & 2,4 & 2,5 & 2,6 \\ 3,1 & 3,2 & 3,3 & 3,4 & 3,5 & 3,6 \\ 4,1 & 4,2 & 4,3 & 4,4 & 4,5 & 4,6 \\ 5,1 & 5,2 & 5,3 & 5,4 & 5,5 & 5,6 \\ 6,1 & 6,2 & 6,3 & 6,4 & 6,5 & 6,6 \end{Bmatrix} \quad \text{... (Q.14.1)}$$

The random variable assigns sum of outcomes of two dice. The minimum outcomes will have a sum of 1 + 1 = 2 and maximum outcomes will have a sum of 6 + 6 =12. Thus the random variable will take values (sums) from 2 to 12 as follows :

$$X = \{x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}\}$$

Now let us find the probabilities of random variable.

$P(X < x_2) = 0$ *i.e. sum less than 2 is not possible.*

$P(X = x_2)$ ***: Sum** = 2 There is only one event having sum of '2'*

i.e. $x_2 = \{1, 1\}$

$$P(X = x_2) = \frac{1}{36}$$

$P(X = x_3)$ **: Sum** = 3 Two events result in sum of '3' i.e.

$x_3 = \{1, 2 ; 2,1\}$

$$\therefore \quad P(X = x_3) = \frac{2}{36}$$

$P(X = x_4) : x_4 = \{1, 3 ; 2, 2 ; 3, 1\}$ i.e., 3 events

$\therefore \quad P(X=x_4) = \frac{3}{36}$

Similarly, $P(X=x_5) = \frac{4}{36}$

$\therefore \quad P(X=x_6) = \frac{5}{36} \quad P(X=x_7) = \frac{6}{36} \quad P(X=x_8) = \frac{5}{36}$

$P(X=x_9) = \frac{4}{36} \quad P(X=x_{10}) = \frac{3}{36} \quad P(X=x_{11}) = \frac{2}{36}$

$P(X=x_{12}) = \frac{1}{36}$

Calculation of CDF

$F_X(x) = 0 \quad \text{for} \quad x < x_2 \text{ since } P(X=x_2) = 0$

$F_X(x_2) = P(X \le x_2) = P(X < x_2) + P(X = x_2) = 0 + \frac{1}{36} = \frac{1}{36}$

$F_X(x_3) = P(X \le x_3) = P(X < x_2) + P(X = x_2) + P(X = x_3)$

$= 0 + \frac{1}{36} + \frac{2}{36} = \frac{3}{36}$

$F_X(x_4) = P(X \le x_4)$

$= P(X < x_2) + P(X = x_2) + P(X = x_3) + P(X = x_4)$

$= 0 + \frac{1}{36} + \frac{2}{36} + \frac{3}{36} = \frac{6}{36}$

Similarly, $F_X(x_5) = \frac{10}{36} \quad F_X(x_6) = \frac{15}{36} \quad F_X(x_7) = \frac{21}{36} \quad F_X(x_8) = \frac{26}{36}$

$F_X(x_9) = \frac{30}{36}, \; F_X(x_{10}) = \frac{33}{36} \quad F_X(x_{11}) = \frac{35}{36} \quad F_X(x_{12}) = \frac{36}{36} = 1$

$F_X(x) = 1 \text{ for all } x > x_{12}$

Q.15 Suppose that a certain random variable has the CDF

$$F_X(x) = \begin{cases} 0 & x \le 0 \\ kx^2 & 0 < x \le 10 \\ 100k & \text{for } x > 10 \end{cases} \quad \text{... (Q.15.1)}$$

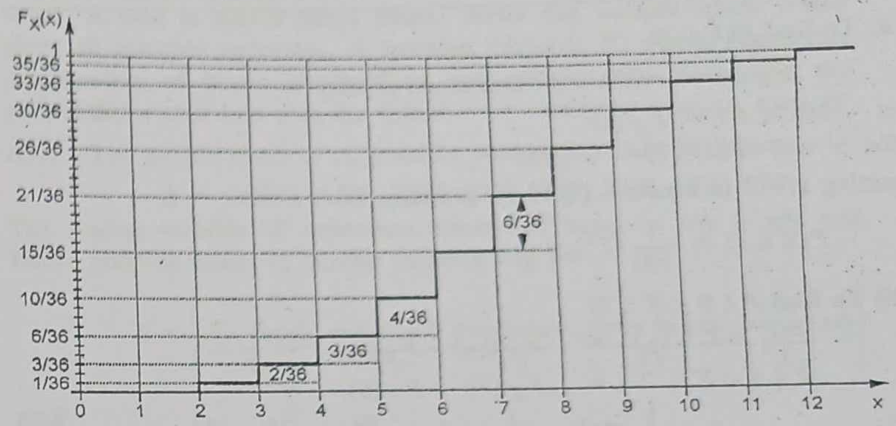


Fig Q.14.1 : CDF associated with rolling of two dice

Evaluate k, Find the values of P(X ≤ 5) and P(5 < X ≤ 7) , plot the corresponding PDF.

[SPPU : May-16,17, Dec.-17,18, Marks 7, Dec.-19, Marks 6]

Ans. : i) To obtain value of 'k'

The CDF can be expressed as,

$$F_X(x) = \begin{cases} 0 & \text{for } x < x_1 \\ \sum_{i=1}^{n} P(x=x_i) & x_1 \le x \le x_n \\ 1 & \text{for } x > x_n \end{cases} \quad \text{... (Q.15.2)}$$

Comparing equation (Q.15.1) and definition of CDF of given above we have,

$F_X(x) = 100\text{ k} = 1 \quad \text{for } x > 10$

$\therefore \quad k = \frac{1}{100}$

Putting the value of k in given equation (Q.15.1),

$$F_X(x) = \frac{1}{100} x^2 \quad \text{for } 0 < x \le 10 \quad \text{... (Q.15.3)}$$

ii) To find $P(X \leq 5)$

$$F_X(x) = P(X \leq x) \quad \text{Definition of CDF}$$

or $P(X \leq x) = F_X(x)$

for $x = 5$, $P(X \leq 5) = F_X(5)$

putting $x = 5$ in equation (Q.15.3). We have,

$$P(X \leq 5) = \frac{1}{100}(5^2) = \frac{1}{4}$$

iii) To find $P(5 < X \leq 7)$

$$P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$$

$$\therefore \quad P(5 < X \leq 7) = F_X(7) - F_X(5)$$

$$= \frac{1}{100}(7)^2 - \frac{1}{100}(5)^2 = \frac{49}{100} - \frac{25}{100} = \frac{24}{100}$$

from equation (Q.15.3)

iv) To find *PDF*

with $k = \frac{1}{100}$, CDF of equation (Q.15.1) becomes,

$$F_X(x) = \begin{cases} 0 & for\, x \leq 0 \\ \frac{x^2}{100} & for\, 0 < x \leq 10 \\ 1 & for\, x > 10 \end{cases} \quad \text{... (Q.15.4)}$$

Now differentiating above equation with respect to x we get PDF as,

$$f_X(x) = \frac{d}{dx}\left[\frac{x^2}{100}\right] = \frac{x}{50} \quad \text{for } 0 < x \leq 10 \quad \text{... (Q.15.5)}$$

Fig. Q.15.1 shows the plot of *PDF* $f_X(x)$ given by above equation,

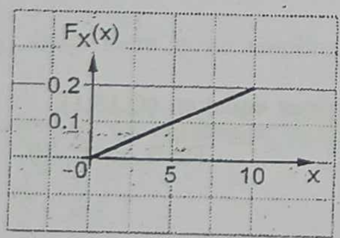


Fig. Q.15.1 *PDF* for a *CDF* given by equation Q.15.5

Q.16 In a random experiment, a trial consists of four successive tosses of a coin. If we define a random variable x as the number of heads appearing in a trial determine PDF and CDF.

☞ [SPPU : Dec.-17, Marks 7]

Ans. : For four successive tosses of coin following is the sample space.

S = {TTTT, TTTH, TTHT, TTHH, THTT, THTH, THHT, THHH, HTTT, HTTH, HTHT, HTHH, HHTT, HHTH, HHHT, HHHH}

$$X = \left\{ \underbrace{\text{No heads}}_{x_0}, \underbrace{\text{one heads}}_{x_1}, \underbrace{\text{two heads}}_{x_2}, \underbrace{\text{three heads}}_{x_3}, \underbrace{\text{four heads}}_{x_4} \right\}$$

$$\therefore \quad P(X = x_0) = \frac{1}{16} \quad P(X = x_3) = \frac{4}{16}$$

$$P(X = x_1) = \frac{4}{16} \quad P(X = x_4) = \frac{1}{16}$$

$$P(X = x_1) = \frac{6}{16}$$

CDF is calculated as follows :

$$F_X(x) = 0 \text{ for } x < x_0$$

$$F_X(x_0) = P(X \leq x_0) = P(X \leq x_0) + P(X \leq x_0) = \frac{1}{16}$$

$$F_X(x_1) = P(X \leq x_1) = P(X \leq x_0) + P(X \leq x_1) = \frac{1}{16} + \frac{4}{16} = \frac{5}{16}$$

$$F_X(x_2) = P(X \leq x_2) = P(X \leq x_1) + P(X \leq x_2) = \frac{5}{16} + \frac{6}{16} = \frac{11}{16}$$

$$F_X(x_3) = P(X \leq x_3) = P(X \leq x_2) + P(X \leq x_3) = \frac{11}{16} + \frac{4}{16} = \frac{15}{16}$$

$$F_X(x_4) = P(X \leq x_4) = P(X \leq x_3) + P(X = x_4) = \frac{15}{16} + \frac{1}{16} = 1$$

PDF is given by,

$$F_X(x) = \frac{d}{dx} f_X(x)$$

$F_X(x_0) = \frac{1}{16}, \; F_X(x_1) = \frac{4}{16}, \; F_X(x_2) = \frac{6}{16}, \; F_X(x_3) = \frac{4}{16}$ and

$F_X(x_4) = \frac{1}{16}$

Q.16 The probability density function of a random variable X is given by : $f_X(X) = e^{-x}u(x)$ determine : i) CDF ii) $P(X \leq 1)$ iii) $P(1 < X \leq 2)$ iv) $P(X > 2)$ ☞ [SPPU : May-18, Marks 7]

Ans. :

i) CDF, $F_X(x) = \int_{-\infty}^{x} F_X(x)\,dx = \int_{-\infty}^{x} e^{-x}\,u(x)\,dx = \int_{0}^{x} e^{-x}dx = \left[\frac{e^{-x}}{-1}\right]_0^x$

$= 1 - e^{-x}$

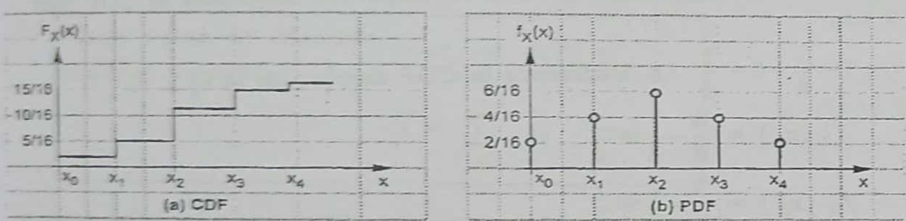


Fig. Q.16.1 : CDF and PDF

ii) $P(X \leq 1) = \int_{0}^{1} F_X(x)\,dx = \int_{0}^{x} e^{-x}dx = \left[\frac{e^{-x}}{-1}\right]_0^1 = 1 - e^{-x}$

iii) $P(1 < X \leq 2) = F_X(2) - F_X(1) \int_{1}^{2} F_X(x)\,dx = \int_{1}^{2} e^{-x}\,dx = \left[\frac{e^{-x}}{-1}\right]_1^2$

$= e^{-1} - e^{-2}$

iv) $P(x > 2) = \int_{2}^{\infty} F_X(x)\,dx = \int_{2}^{\infty} e^{-x}dx = \left[\frac{e^{-x}}{-1}\right]_2^{\infty} = e^{-2}$

Q.17 A coin is tossed three times. Write the sample space which gives all possible outcomes. A random variable X, which represents the number of heads obtained on any tripple toss. Also, find the probabilities of X and plot the C.D.F. ☞ [SPPU : Dec.-18, Marks 6]

Ans. : The sample space of all possible outcomes will be as follows :

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

The random variable 'X' represents number of heads in any tripple toss. Hence possible values of random variable will be,

$$X = \left\{\underbrace{No\ heads}_{x_0}, \underbrace{one\ heads}_{x_1}, \underbrace{\text{two heads}}_{x_2}, \underbrace{three\ heads}_{x_3}, \underbrace{four\ heads}_{x_4}\right\}$$

CDF

The CDF is given as,

$$\therefore \quad F_X(x) = \begin{cases} 0 & \text{for} \quad x < x_0 \\ \sum_{i=1}^{2} P(X = x_i) & \text{for } x_0 \leq x_3 \\ 1 & \text{for} \quad x > x_3 \end{cases}$$

$$= P(X = x_0) + P(X = x_1) + P(X + x_2) + P(X + x_3)$$

$P(X = x_0)$: Probability of getting no heads : In sample space there is only one outcome that has all tails. i.e.,

$$x_0 = \{T, T, T\}$$

Probability of getting head or tail is $\frac{1}{2}$. Hence,

$$P(X = x_0) = P(T) \cdot P(T) \cdot P(T) \times \text{Number of outcomes for } x_0$$
$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times 1 = \frac{1}{8}$$

$P(X = x_1)$: Probability of getting one head : In sample space there are three outcomes that has one head i.e.,

$$x_1 = \{HTT, THT, TTH\}$$

$$\therefore \quad P(X = x_1) = P(H) \cdot P(T) \cdot P(T) \times \text{Number of outcomes for } x_1$$
$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times 3 = \frac{3}{8}$$

$P(X = x_2)$: Probability of getting two heads : In sample space there are 3 outcomes that has two heads. i.e.,

$$x_2 = \{HHT, HTH, THH\}$$

$$= P(H) \cdot P(H) \cdot P(T) \times \text{Number of outcomes for } x_2$$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times 3 = \frac{3}{8}$$

$P(X = x_3)$: Prob.bility of getting all heads : In sample space only one outcome has all heads. i.e.,

$$x_3 = \{HHH\}$$

$$\therefore \quad P(X = x_3) = P(H) \cdot P(H) \cdot P(H) \times \text{Number of outcomes for } x_3$$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times 1 = \frac{1}{8}$$

Now CDF can be calculated as follows :

$$F_X(x) = 0 \qquad \text{For } x < x_0$$

$$F_X(x_0) = P(X \le x_0) = P(X < x_0) + P(X = x_0)$$

$$= 0 + \frac{1}{8} = \frac{1}{8}, \text{ since } P(X < x_0) = 0$$

$$F_X(x_1) = P(X \le x_1) = P(X < x_0) + P(X = x_0) + P(X = x_1)$$

$$= 0 + \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$$

Similarly, $F_X(x_2) = P(X \le x_2)$

$$= P(X \le x_0) + P(X < x_0) + P(X = x_1) + P(X = x_2)$$

$$= 0 + \frac{1}{8} + \frac{3}{8} + \frac{3}{8} = \frac{7}{8}$$

$$F_X(x_3) = P(X \le x_3)$$

$$= P(X \le x_0) + P(X < x_0) + P(X = x_1) + P(X = x_2) + P(X = x_3)$$

$$= 0 + \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$$

Fig. Q.17.1 (a) shows the CDF of X.

PDF $f_X(x) = \frac{d}{dx} F_X(x)$ i.e. Difference in amplitudes of CDF plot of Fig. Q.17.1.

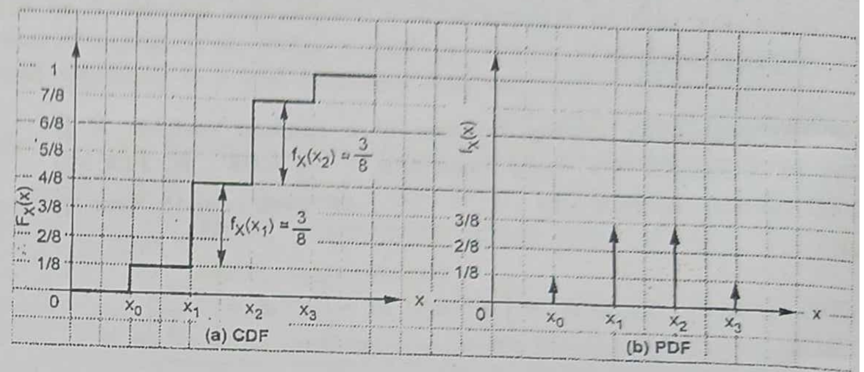


Fig. Q.17.1 : Plots of CDF and PDFs

$f_X(x) = 0$ for $x < x_0$, since there is no amplitude difference in CDF plot for $x < x_0$

$$f_X(x_0) = \frac{1}{8} - 0 = \frac{1}{8} \quad \text{i.e. Difference in CDF amplitudes at } x_0$$

$$f_X(x_1) = \frac{4}{8} - \frac{1}{8} = \frac{3}{8}$$

$$f_X(x_2) = \frac{7}{8} - \frac{4}{8} = \frac{3}{8}$$

$$f_X(x_3) = 1 - \frac{7}{8} = \frac{1}{8}$$

$f_X(x) = 0$ for $x > x_3$ since there is no amplitude difference in CDF plot for $x > x_3$.

Fig. Q.17.1 (b) shows the PDF calculated above.

6.4 : Statistical Averages

Important Points to Remember

- The mean of the random variable is equal to summation of the values of 'X' weighted by their probabilities.
- The n[th] moment of a random variable 'X' is equal to mean value of X^n.

- First moment is mean value, second moment is mean square value and second central moment is variance.
- n^{th} control moment, $E[(X - m_x)^n] = \int_{-\infty}^{\infty} (x - m_x)^n f_X(x) dx.$

Q.18 What is mean/average or expected value of a random variable? How it is calculated for continuous and discrete random variables ?

☞ [SPPU : Dec.-07, 11, Marks 2, Dec.-08, May-07, June-22, Marks 3]

Ans. : • Definition : The mean of the random variable is given by summation of the values of X weighted by their probabilities.

Mean value is denoted by m_x. Mean value is also called expected value of X. Therefore,

$$m_x = E[X] \quad \ldots (Q.18.1)$$

• The mean value of discrete random variable is given as,

$$\text{Mean value of discrete random variable} = m_x = E[X] = \overline{X} = \sum_{i=1}^{n} x_i \, P(x_i) \quad \ldots (Q.18.2)$$

• Mean value of a continuous random variable is given as,

$$\text{Mean value of continuous random variable} = m_x = \int_{-\infty}^{\infty} x \, f_X(x) \, dx \quad \ldots (Q.18.3)$$

Q.19 Define the terms : Moments, expection, variance, mean square value and standard deviation.

☞ [SPPU : Dec.-07, 11, Marks 2, May-12, Marks 6, Dec.-14, Marks 4, June-22, Marks 3]

Ans. : Expectation or mean value is defined in Q.18.

Moments : The n^{th} moment of a random variable 'X' is defined as mean value of X^n. i.e.,

$$\overline{X^n} = E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x) \, dx$$

Mean Square Value : The second moment is called mean square value. i.e.,

$$\overline{X^2} = E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) \, dx$$

Variance : The second central moment is called variance of random variable i.e.,

$$\sigma_x^2 = \text{Var}[X] = E[(X - m_x)^2] = \int_{-\infty}^{\infty} (x - m_x)^2 f_X(x) \, dx$$

$$= \overline{X^2} - m_x^2$$

Standard deviation : It gives the spread of values of a random variable 'X' from its mean value.

$$\sigma_x = \sqrt{\sigma_x^2} = \sqrt{\overline{X^2} - m_x^2}$$

Q.20 The PDF of a random variable x is given by :

$$f_x(x) = 1/2\pi \quad \text{for } 0 \le x \le 2\pi$$
$$= 0 \quad \text{otherwise}$$

Calculate mean value, mean square value, variance and standard derivation. ☞ [SPPU : May-17, Marks 7]

Ans. : i) Mean value

$$m_x = \int_{-\infty}^{\infty} x \, f_x(x) \, dx = \int_{0}^{2\pi} x \cdot \frac{1}{2\pi} dx$$

$$= \int_{0}^{2\pi} x \cdot \frac{1}{2\pi} dx$$

$$= \frac{1}{2\pi} \left[\frac{x^2}{2} \right]_0^{2\pi}$$

$$= \frac{1}{2\pi} \frac{(2\pi)^2 - 0}{2} = \pi$$

ii) Mean square value

$$E[X^2] = \int_{-\infty}^{\infty} x^2 \cdot f_X(x)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x^2\,dx$$

$$= \frac{1}{2\pi}\cdot\left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{2\pi}\cdot\frac{(2\pi)^3-0}{3} = \frac{4\pi^2}{3}$$

iii) Variance σ_x^2

$$\sigma_x^2 = E[X^2] - m_x^2$$

$$= \frac{4\pi^2}{3} - \pi^2 = \frac{\pi^2}{3}$$

iv) Standard deviation

$$\sigma_x = \sqrt{\frac{\pi^2}{3}} = \frac{\pi}{\sqrt{3}}$$

Q.21 A random variable X is $f_X(x) = 5x^2 \quad 0 \le x \le 1$

$= 0 \;;\; elsewhere$

Find E [X], E [3X – 2], E [X^2].

☞[SPPU : May-15, Marks 6, May-19, Marks 4]

Ans. : i) *E [X]* or mean value :

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 x\cdot 5x^2\,dx = \int_0^1 5x^3\,dx = 5\left[\frac{x^4}{4}\right]_0^1 = \frac{5}{4}$$

ii) *E [3X–2]* :

Let, $g[X] = 3x-2$

$$E[g(x)] = \int_{-\infty}^{\infty} g(X) f_X(x)\,dx = \int_0^1 (3x-2)\cdot 5x^2\,dx$$

$$= 15\int_0^1 x^3\,dx - 10\int_0^1 x^2\,dx = 15\left[\frac{x^4}{4}\right]_0^1 - 10\left[\frac{x^3}{3}\right]_0^1 = \frac{5}{12}$$

ii) $E[X^2]$:

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_0^1 x^2\, 5x^2\,dx = 5\int_0^1 x^4\,dx = 5\cdot\left[\frac{x^5}{5}\right]_0^1 = 1.$$

Q.22 Find the mean, second moment and standard deviation of 'X' when $F_X(x) = Ae^{-Ax}u(x)$. **☞[SPPU : Dec.-11, Marks 6, Dec.-19, Marks 3]**

Ans. : Here, $f_x(x) = A_e^{-Ax} u(x)$

i) Mean E[X] :

$$m_x = E[X] = \int_{-\infty}^{\infty} x f_x(x)\,dx = \int_0^{\infty} x A e^{-Ax}\,dx$$

Since, $u(x) = 0$ for $x < 0$

$$= A\int_0^{\infty} x e^{-Ax}\,dx$$

Integrating by parts,

$$m_x = E[X] = A\left[x\cdot\frac{e^{-Ax}}{-A} - \int\frac{e^{-Ax}}{-A}\cdot 1\,dx\right]_0^{\infty} = \frac{1}{A}$$

ii) Second moment : n^{th} moment is given as,

$$\overline{X^n} = E[X^n] = \int_{-\infty}^{\infty} x^n f_x(x)\,dx$$

$$n = 2,\quad \overline{X^2} = E[X^2] = \int_{-\infty}^{\infty} x^2 f_x(x)\,dx$$

$$= \int_0^{\infty} x^2 Ae^{-Ax}\,dx, \text{ since } u(x) = 0 \text{ for } x < 0$$

$$= A\int_0^{\infty} x^2 e^{-ax}\,dx$$

Integrating by parts,

$$\overline{X^2} = E[X^2] = A\left[x^2 \cdot \frac{e^{-Ax}}{-A} - \int \frac{e^{-Ax}}{-A} \cdot 2x\,dx\right]_0^{\infty}$$

$$= [-x^2 e^{-Ax}]_0^{\infty} + 2\int_0^{\infty} x e^{-Ax}\,dx$$

In the above equation, first term will be zero after putting limits. Let us, rearrange second term as follows,

$$\overline{X^2} = E[X^2] = \frac{2}{A} \cdot \left[A \int_0^{\infty} x e^{-Ax}\,dx\right]$$

Note that the term inside bracket is mean value, m_x. Putting its value from part (i),

$$\overline{X^2} = E[X^2] = \frac{2}{A} \cdot \frac{1}{A} = \frac{2}{A^2}$$

iii) Standard deviation :

$$\delta_x = \sqrt{E[X^2] - m_x^2} = \sqrt{\frac{2}{A^2} - \frac{1}{A^2}} = \frac{1}{A}$$

Q.23 The probability density function of a random variable 'X' is given by : $f_X(x) = \begin{cases} \frac{1}{a} & |x| \le a \\ 0 & otherwise \end{cases}$

Determine : i) Mean $E[X]$ ii) Mean square value $E[X^2]$ iii) Standard deviation. ☞ [SPPU : May-18, Marks 7, Dec.-22, Marks 6]

Ans. : i) Mean :

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_{-a}^{a} x \frac{1}{a}\,dx = \frac{1}{a}\left[\frac{x^2}{2}\right]_{-a}^{a} = 0$$

ii) Mean square value :

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_{-a}^{a} x^2 \cdot \frac{1}{a}\,dx = \frac{1}{a}\left[\frac{x^3}{3}\right]_{-a}^{a} = \frac{2a^2}{3}$$

iii) Standard deviation

$$\sigma_x = \sqrt{E[X^2] - E(X)}$$

$$= \sqrt{\frac{2a^2}{3} - 0} = a\sqrt{\frac{2}{3}}$$

FORMULAE AT A GLANCE

i) Probability, $P_A = \lim_{N \to \infty} \frac{N_A}{N}$

$P(AB) = P(A) \cdot P(B/A) = P(B) \cdot P(A/B)$

$$P = (B_i / A) = \frac{P(B_i) P(A/B_i)}{\sum_{i=1}^{n} P(B_i) P(A/B_i)} \quad \text{... (6.3)}$$

ii) Binomial distribution $m_x = np$,

$\sigma_x^2 = np\,(1-p)$

iii) Poisson distribution, $m_x = np$, $\sigma_x^2 = np$

iv) Uniform distribution, $m_x = \frac{a+b}{2}$, $\sigma_x^2 = \frac{(a-b)^2}{12}$

v) Gaussian distribution, $m_x = m$, $\sigma_x^2 = \sigma^2$

vi) CDF, $F_X(x) = P(X \le x)$

PDF, $f_X(x) = \frac{d}{dx} F_X(x)$

vii) Statistical averages, $m_x = E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$

$$\overline{X^2} = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$$

$$\sigma_x^2 = \overline{X^2} - m_x^2$$

END...

DECEMBER - 2016 [5057] - 2042 Solved Paper

Time : Two Hours] [Maximum Marks : 50

Instructions to the Candidates :

1) Attempt four questions Q. 1 or Q. 2, Q. 3 or Q. 4, Q. 5 or Q. 6 and Q. 7 or Q. 8.

2) Figures to the right side indicate full marks.

3) Assume suitable aata, if necessary.

4) Neat diagrams must be drawn wherever necessary.

5) Use of electronic non-programmable Calculator is allowed.

Q.1 a) *Find whether the following signals are energy or power and find the corresponding value :* $x(t) = (1/2)^2 \cdot u[n]$

[Refer Q.24 (ii) of Chapter - 1] [4]

b) *Find the convolution between :* $x[n] = \{1, 1, 1, 1\}$ *and* $h(n) = \{1, 1, 1, 1\}$. **[Refer Q.11 of Chapter - 2]** [4]

c) *Find odd and even components of the signal :* [4]

$x[n] = u[n] - u[n-4]$ **[Refer Q.21 of Chapter - 1]**

OR

Q.2 a) *An analog signal is given by the equation :*

$x(t) = 2 \sin 400 \pi t + 10 \cos 1000 \pi t$.

It is sampled at sampling frequency 1000 Hz.

i) What is the Nyquist rate for the above signal ?

ii) What is the Nyquist interval of the signal ? [2]

Ans. : Here $f_s = 1000$ Hz.

Compare given equation with

$$x(t) = A_1 \sin 2\pi f_1 t + A_2 \cos 2\pi f_2 t$$

$$\therefore \quad f_1 = 200 \text{ Hz and } f_2 = 500 \text{ Hz}$$

Here highest frequency is,

$$W = f_2 = 500 \text{ Hz}$$

i) Nyquist rate = 2 W = 2 × 500 = 1000 Hz.

ii) Nyquist interval $= \frac{1}{2W} = \frac{1}{1000} = 0.001$ sec.

b) *Find the convolution between :* $x(t) = u(t)$ *and* $h(t) = u(t-2)$ *using graphical method.* **[Refer Q.4 of Chapter - 2]** [6]

c) *Check whether the following signal is periodic or non-periodic. If periodic, find period of the signal :* $x(t) = \cos(n/8).\cos(n\pi/8)$. [4]

Ans. : Compare given equation with

$$x(n) = \cos(2\pi f_1 n) \cdot \cos(2\pi f_2 n)$$

$$\therefore \quad f_1 = \frac{\pi}{4} \text{ and } f_2 = \frac{1}{4}$$

Here, $f_1 = \frac{\pi}{4}$ is not the ratio of two integers, since π is not an integer. But f_2 is ratio of two integers. Hence $\cos(n/8)$ is non periodic but $\cos(n\pi/8)$ is periodic. The product of periodic and nonperiodic signal is also **nonperiodic.**

Q.3 a) *State and explain the properties of continuous time fourier series.* **[Refer Q.7 of Chapter - 3]** [6]

b) *Determine the transfer function and impulse response for the system described by the differential equation shown below for zero initial conditions :* $d/dt\,[y(t)] + 3\,y(t) = x(t)$ **[Refer Q.19 of Chapter - 4]** [6]

OR

Q.4 a) *Draw the magnitude and phase spectrum of the signal :* $x(t) = 5\cos(2\pi 10t + 30) - 10\cos(2\pi 20t + 60)$ [6]

Ans. : We know that, $\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$.

Hence given equation can be written as,

$$x(t) = 5 \cdot \frac{e^{j(2\pi 10t+30)} + e^{-j(2\pi 10t+30)}}{2} - 10 \cdot \frac{e^{j(2\pi 20t+60)} + e^{-j(2\pi 20t+60)}}{2}$$

$$= 2.5\, e^{j2\pi 10t} \cdot e^{j30°} + 2.5\, e^{j2\pi 10t} \cdot e^{-j30°}$$
$$-5e^{j2\pi 20t} \cdot e^{j60°} - 5e^{-j2\pi 20t} \cdot e^{-j60°}$$

Here note that one of the frequency of cosine wave is 10 Hz. and other frequency is 20 Hz. Hence fundamental frequency is $f_o = 10$ Hz.

$$\therefore \quad x(t) = A_1\, e^{j2\pi f_0 t} \cdot e^{j\phi_1} + A_1\, e^{-j2\pi f_0 t} \cdot e^{-j\phi_1}$$
$$+ A_2\, e^{j2\pi f_0 t} e^{j\phi_2} + A_2 e^{-j2\pi f_0 t} e^{-j\phi_2}$$

Here $A_1 = 2.5$ V, $f_0 = 10$ Hz, $\phi_1 = 30°$ and

$A_2 = -5$ V, $\phi_2 = 60°$.

Fig. 1 shows the magnitude and phase spectrum based on above values.

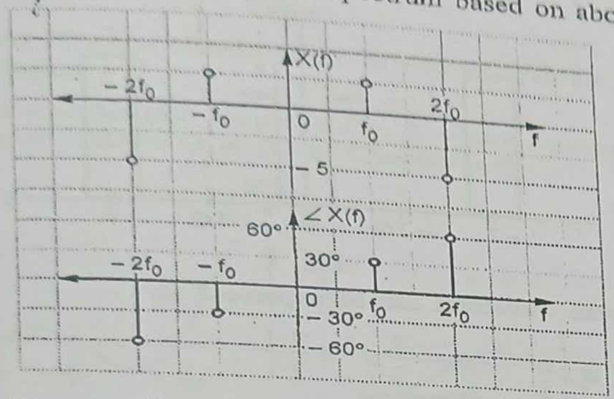


Fig. 1 : Magnitude and phase spectrum

b) *Find the Fourier transform of the signal* $x(t) = \sin \omega_c t \cdot u(t)$ **[Refer Q.5 (i) of Chapter - 4]** [6]

Q.5 a) *State and prove convolution property of Laplace transform.* **[Refer Q.7 of Chapter - 5]** [6]

b) *Find the initial and final value of :* $X(s) = (5s + 50)/s(s+5)$ [7]

Ans. : Initial value is given as,

$$x(0+) = \lim_{s \to \infty} s\, X(s) = \lim_{s \to \infty} \frac{5s+50}{s+5} = \lim_{s \to \infty} \frac{5+\frac{50}{s}}{1+\frac{5}{s}} = 5$$

Final value is given as,

$$\lim_{t \to \infty} x(t) = \lim_{s \to \infty} s\, X(s) = \lim_{s \to 0} \frac{5s+50}{s+5} = 10$$

OR

Q.6 a) *Find the Laplace transform of the given signal and draw its ROC :* $x(t) = -e^{at} u(-t)$ **[Refer Q.9 (ii) of Chapter - 5]** [6]

b) *Find the inverse Laplace transform of :* [7]

$X(s) = (3s+7)/(s^2 - 2s - 3)$ **[Refer Q.31 of Chapter - 5]**

Q.7 a) *List the properties of auto correlation and cross correlation for energy signals.* **[Not in New Syllabus]** [6]

b) *A perfect die is thrown. Find the probability that :* [7]

i) You get even number ii) You get a perfect square.

Ans. : Sample space for throwing a perfect die will be,

S = {1, 2, 3, 4, 5, 6} Hence N = 6 samples

i) Getting even number, A = {2, 4, 6} Hence $N_A = 3$

$$\therefore \quad P(A) = \frac{N_A}{N} = \frac{3}{6} = 0.5$$

ii) Getting a perfect square, B = {1, 4}, Hence N = 2

$$\therefore \quad P(B) = \frac{N_B}{N} = \frac{2}{6} = 0.333$$

OR

Q.8 a) *List the properties of probability. Explain conditional probability with an example and formula.* [6]

Ans. : 1. Probability of any event is always less than or equal to '1' and non-negative. $0 \le P(A) \le 1$

2. If $A + B$ is the union of two mutually exclusive events then,

$$P(A + B) = P(A) + P(B)$$

3. If $\overline{A}$ denotes the compliment of event A, $P(\overline{A}) = 1 - P(A)$

4. If $A_1, A_2, \ldots A_M$ are mutually exclusive events,

$P(A_1) + P(A_2) + \ldots + P(A_M) = 1$

5. If A and B are not mutually exclusive, then

$$P(A + B) = P(A) + P(B) - P(AB)$$

6. Conditional probabilities are given as,

$$P(B/A) = \frac{P(AB)}{P(A)} \text{ and } P(A/B) = \frac{P(AB)}{P(B)}$$

7. If A and B are statistically independent,

$$P(AB) = P(A) \cdot P(B)$$

b) *A three digit message is transmitted over a noisy channel having a probability of error as P(E) = 2/5 per digit. Find and draw the CDF.* **[Refer Q.12 of Chapter - 6]** [7]

MAY - 2017 [5152] - 531 — Solved Paper

Time : Two Hours] [Maximum Marks : 50

Q.1 a) *Find whether the following signals are energy or power and find the corresponding value $x(t) = \cos(t)$* **[Refer Q.23 of Chapter - 1][4]**

b) *Determine whether the following LTI system described by impulse responce $h(t) = e^{-t}u(t+1)$ is stable and casual.* **[4]**

[Refer Q.19 of Chapter - 2]

c) *Find odd and even components of the following signals :* **[4]**

$x[n] = \{1, 0, -1, 2, 3\}$ **[Refer Q.22 of Chapter - 1]**

OR

Q.2 a) *An analog signal is given by the equation :*

$x(t) = 2\sin 400\pi t + 10\cos 1000\,\pi t$

It is sampled at sampling frequency 1000 Hz :

i) What is the Nyquist rate for the above signal ?

ii) What is the Nyquist interval of the signal ? **[Not in New Syllabus] [2]**

b) *Determine the convolution sum of the following sequence using equation of convolution sum : $x(n) = \delta(n) + 2\delta(n-2)$, $h(n) = 2\delta(n) + \delta(n-2)$.* **[Refer Q.10 of Chapter-2]** **[6]**

c) *Check whether the following signal is periodic or non-periodic. If periodic, find period of the signal : $x(t) = 10\sin 12\pi t + 4\sin 18\pi t$* **[4]**

Ans. : Compare given equation with $x(t) = A_1\sin(2\pi f_1 t) + A_2\sin(2\pi f_2 t)$

$\therefore \quad 2\pi f_1 = 12\pi \Rightarrow f_1 = 6$, hence $T_1 = \frac{1}{f_1} = \frac{1}{6}$

and $\quad 2\pi f_2 = 18\pi \Rightarrow f_2 = 9$, hence $T_2 = \frac{1}{f_2} = \frac{1}{9}$

Since $\frac{T_1}{T_2} = \frac{16}{19} = \frac{3}{2}$, which is rational. Hence the signal is **periodic**. The fundamental period will be, $T = 2T_1 = 3T_2 = 2 \times \frac{1}{6} = \frac{1}{3}$ sec.

Q.3 a) *State and prove the following properties of CTFT :*

i) Time scaling ii) Time shifting **[Refer Q.9 of Chapter - 4]** **[6]**

b) *Obtain the trigonometric Fourier series of the rectangular pul shown in Fig. 1 :*

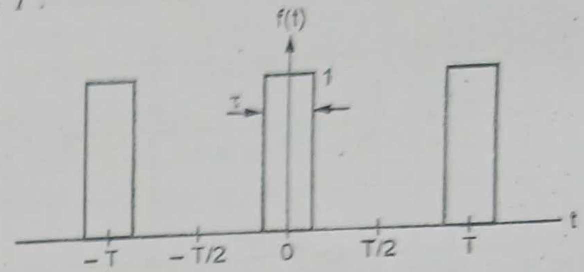


Fig. 1

Ans. : The given waveform can be expressed as,

$$f(t) = \begin{cases} 1 & for -\frac{\tau}{2} \le t \le \frac{\tau}{2} \\ 0 & elsewhere \end{cases}$$

Period = T Hence $= \omega_0 = \frac{2\pi}{T}$

$$a(0) = \frac{1}{T}\int_{<T>} f(t)\,dt = \frac{1}{T}\int_{-\tau/2}^{\tau/2} 1\,dt = \frac{\tau}{T}$$

$$a(k) = \frac{2}{T}\int_{<T>} f(t)\cos(k\omega_0 t)\,dt = \frac{2}{T}\int_{-\tau/2}^{\tau/2} 1\cdot\cos(k\omega_0 t)\,dt$$

$$= \frac{2}{T}\left[\frac{\sin k\omega_0 t}{k\omega_0}\right]_{-\tau/2}^{\tau/2} = \frac{4}{k\omega_0 T}\sin\frac{k\omega_0\tau}{2} = \frac{2}{k\pi}\sin\frac{k\omega_0\tau}{2}$$

since, $\omega_0\,\tau = 2\pi$

$$b(k) = \frac{2}{T}\int_{<T>} f(t)\sin k\omega_0 t\,dt$$

$$= \frac{2}{T}\int_{-\tau/2}^{\tau/2} 1\cdot\sin k\omega_0 t\,dt = \frac{2}{T}\left[-\frac{\cos k\omega_0 t}{k\omega_0}\right]_{-\tau/2}^{\tau/2} = 0$$

Hence trigonometric Fourier series will be,

$$f(t) = a(0) + \sum_{k=1}^{\infty} a(k)\cos k\omega_0 t + \sum_{k=1}^{\infty} b(k)\sin k\omega_0 t$$

$$= \frac{\tau}{T} + \sum_{k=1}^{\infty} \frac{2}{k\pi} \sin \frac{k\omega_0 \tau}{2} \cos(k\omega_0 t)$$

OR

Q.4 a) *State the Dirichlet conditions for existence of fourier series.* **[Refer Q.1 of Chapter - 3]** [4]

b) *For the sinc function shown in Fig. 2. obtain Fourier transform and plot it spectrum :* **[Refer Q.14 (ii) of Chapter - 4]** [8]

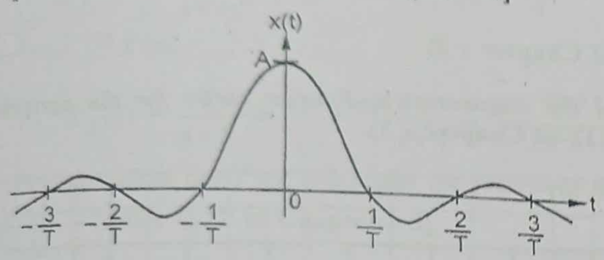


Fig. 2

Q.5 a) *Find the initial and final value of a signal :* [6]

$X(s) = (s+10)/(s^2+2s+2)$ **[Refer Q.24 of Chapter - 5]**

b) *Find the inverse Laplace transform of :* [7]

$X(s) = -5s - 7/(s+1)(s-1)(s+2)$

[Refer Q.29 of Chapter - 5]

OR

Q.6 a) *Find the Laplace transform of the following with ROC :* [7]

i) $x(t) = u(t-5)$ **[Refer Q.16 of Chapter - 5]**

ii) $x(t) = e^{-at} \sin(\omega t) u(t)$

Ans. :

ii) $x(t) = e^{-at} \sin \omega t \, u(t)$

Here $x(t) = e^{-at} \sin \omega t \, u(t) = e^{-at} \dfrac{e^{j\omega t} - e^{-j\omega t}}{2j} u(t)$

since $\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j} = \dfrac{1}{2j}[e^{-(a-j\omega)} - e^{-(a+j\omega)t}]\, u(t)$

since $e^{at} u(t) \xleftrightarrow{L} \frac{1}{s-a}$, $ROC : Re(s) > a$.

Laplace transform of above equation becomes,

$$X(s) = \frac{1}{2j}\left\{\frac{1}{s+(a-j\omega)} - \frac{1}{s+(a+j\omega)}\right\} \; ROC : Re(s) > -a$$

$$= \frac{\omega}{(s+a)^2+\omega^2}, \; ROC : \sigma > -a$$

b) *The differential equation of the system is given by :* [6]

$$\frac{dy}{dt}(t) + 2y(t) = x(t)$$

Determine the output of system for $x(t) = e^{-3t}u(t)$. Assume zero initial condition. **[Refer Q.35 of Chapter - 5]**

Q.7 a) *What is correlation ? Explain the two types of correlations with a practical application for each.* **[Not in New Syllabus]** [6]

b) *The PDF of a random variable x is given by :* [7]

$f_X(x) = 1/2\pi$ for $0 \le x \le 2\pi$

$= 0$ otherwise

Calculate mean value, mean square value, variance and standard derivation.

Ans. : $m_x = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^{2\pi} x \frac{1}{2\pi} dx = \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = \pi$

$$\overline{X^2} = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_0^{2\pi} x^2 \cdot \frac{1}{2\pi} dx = \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{4\pi^2}{3}$$

$$\sigma_x^2 = \overline{X^2} - m_x^2 = \frac{4\pi^2}{3} - \pi^2 = \frac{\pi^2}{3}$$

$$\sigma_x = \sqrt{\frac{\pi^2}{3}} = \frac{\sqrt{\pi}}{3}$$

OR

Q.8 a) *In a pack of cards, 2 cards are drawn simultaneously. What is the probability of getting a queen, jack combination ?* [6]

Ans. : Two cards can be drawn in $^{52}C_2$ ways. i.e.,

$$N = {}^{52}C_2 = \frac{52!}{(52-2)!2!} = \frac{52\times 51\times 50!}{50!\times 2\times 1} = 1326 \text{ ways}$$

These are 4 queens and 4 jacks. Hence queen and jack can be chosen in $^4C_1$ ways.

$$\therefore \quad N_A = {}^4C_1 \times {}^4C_1 = \frac{4!}{(4-1)!\times 1!}\times\frac{4!}{(4-1)!\times 1!} = 16$$

$$\therefore \quad \text{Probability} = \frac{N_A}{N} = \frac{16}{1326} = 0.0118$$

b) *Suppose that a certain random variable has a CDF :* **[7]**

$$F_x(X) = 0 \quad \text{for } x \le 0$$
$$= kx^2 \quad \text{for } 0 \le x \le 10$$
$$= 50k \quad \text{for } x > 10$$

i) Determine the value of k *ii)* $P(4 \le x \le 7)$

iii) Find and sketch PDF **[Refer Q.15 of Chapter - 6]**

DECEMBER - 2017 [5252] - 531 | Solved Paper

Course 2015

Time : Two Hours] [Maximum Marks : 50

Q.1 a) *Sketch the following signals :*

i) $u[n+2] - u[n-3]$ *ii)* $r(t)\,u(2-t)$ **(Refer Q.11 of Chapter - 1)** **[6]**

b) *Find the convolution of x(t) and h(t) :*

$x(t) = u(t+1)$

$h(t) = u(t-2)$ **(Refer Q.5 of Chapter - 2)** **[6]**

OR

Q.2 a) *Check whether the following system is static/dynamic, linear/non-linear, causal/non-causal, time variant/time invariant :* **[4]**

$y(t) = 10x(t) + 5$ **(Refer Q.37 of Chapter - 1)**

b) *Check whether the following signal is periodic or non-periodic. If periodic, find the fundamental time period.* **[2]**

$x(t) = 2\cos(10t + 1) - \sin(4t - 1)$

(Refer Q.28 of Chapter - 1)

c) *Determine the convolution sum of two sequences graphically:* **[6]**

$x[n] = \{\underset{\uparrow}{1}, 2\ 3\ 2\}$ $h[n] = \{\underset{\uparrow}{1}, 2\ 2\}$

(Refer Q.12 of Chapter - 2)

Q.3 a) *Find the trigonometric Fourier series for the periodic signal x(t).* **(Refer Q.12 of Chapter - 3)** **[6]**

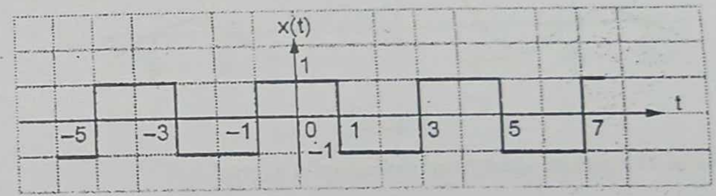


Fig. 1

b) *Obtain the Fourier transform of a rectangular pulse :* **[6]**

$x(t) = A\ rect\ (t/T)$ **(Refer Q.14 of Chapter - 4)**

OR

Q.4 a) *Obtain the exponential Fourier series of the unit impulse train*

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0)$$

Sketch the Fourier spectrum **(Refer Q.14 of Chapter - 3)** **[6**

b) *Find the Fourier transform of the following signals :*

i) $x(t) = \delta(t)$ **(Refer Q.4 (i) of Chapter - 4)**

ii) $x(t) = e^{-at}u(t)$ **(Refer Q.3 (i) of Chapter - 4)** **[**

Q.5 a) *Find the Laplace transform of :*

$x(t) = e^{-5t}[u(t) - u(t-5)]$ *and its ROC*

Ans. : $x_2(t) = e^{-5t}[u(t) - u(t-5)]$

$$= e^{-5t}u(t) - e^{-5t}u(t-5)$$

Let us rearrange this equation as,

$$x_3(t) = e^{-5t}\,u(t) - e^{-5(t-5+5)}\,u(t-5)$$
$$= e^{-5t}\,u(t) - e^{-5(t-5)} \cdot e^{-25}\,u(t-5)$$
$$= e^{-5t}\,u(t) - e^{-25} \cdot e^{-5(t-5)}\,u(t-5)$$

Here use $e^{-at}u(t) \overset{L}{\longleftrightarrow} \frac{1}{s+a}$ and time shift property :

$x(t-t_0) \overset{L}{\longleftrightarrow} e^{-st_0}X(s)$.

$$\therefore \quad X_3(s) = \frac{1}{s+5} - e^{-25} \cdot e^{-5s} \cdot \frac{1}{s+5} = \frac{1-e^{-5(s+5)}}{s+5}$$

b) *Find the initial and final values for the following function :*

$x(s) = \frac{s+5}{s^2+3s+2}$ **(Refer Q.27 of Chapter - 5)** [6]

OR

Q.6 a) *Determine the inverse Laplace transform of :* [7]

$X(s) = \frac{2}{s(s+1)(s+2)}$ **(Refer Q.30 of Chapter - 5)**

b) *Find Laplace transform of given periodic signal :*

(Refer Q.15 of Chapter - 5) [6]

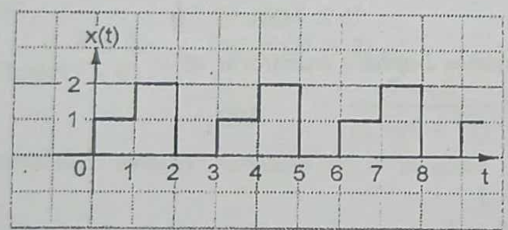


Fig. 2

Q.7 a) *In a random experiment, a trial consists of four successive tosses of a coin. If we define a random variable x as the number of heads appearing in a trial determine PDF and CDF.*

(Refer Q.16 of Chapter - 6) [7]

b) *State and prove any three properties of PDF.*

(Refer Q.11 of Chapter - 6) [6]

OR

Q.8 a) *A certain random variable has the CDF given by :*

$$F_X(x) = 0 \quad for\ x \le 0$$
$$= kx^2 \quad for\ 0 < x \le 10$$
$$= 100k \quad for\ x > 10$$

Find the value of :

i) k *ii) $P(x \le 5)$* *iii) $P(5 < x \le 7)$*

iv) Plot the corresponding PDF **(Refer Q.15 of Chapter - 6)** [7]

b) *State and explain the properties of auto-correlation function for energy signal.* **(Not in New Syllabus)** [6]

MAY - 2018 [5352] - 531 — Solved Paper

Course 2015

Time : Two Hours] **[Maximum Marks : 50**

Q.1 a) *Perform the following operations and sketch the signals :* [6]

(i) $y(t) = r(t+1) - r(t) + u(t-2)$

(ii) $y(n) = u(n+3) - 2u(n-1) + u(n-4)$

(Refer Q.12 of Chapter - 1)

b) *Using impulse response properties, determine whether the following systems are :*

i) *Static/Dynamic*

ii) *Causal/Non-Causal*

iii) *Stable/Unstable :*

1) $h(t) = e^{-2|t|}$

2) $h(n) = 2\delta(n) - 3\delta(n-1)$ **(Refer Q.20 of Chapter - 2)** [6]

OR

Q.2 a) *Find even and odd components of the following signals :*

(i) $x(t) = 3t + t\cos t + t^2 \sin^2 4t$

(ii) $x(n) = \{1, 1, -1, -1\}$ **(Refer Q.18 of Chapter - 1)** [6]

b) *Find convolution of the following, using graphical method :*

(i) $x(n) = u(n)$

(ii) $h(n) = a^n u(n) \quad 0 < a < 1.$

(Refer Q.18 of Chapter - 2) **[6]**

Q.3 a) *Find Fourier transform of the following signals using appropriate properties :*

(i) $x(t) = \frac{d}{dt}\{e^{-at} \cdot u(t)\}$ **(Refer Q.17 (i) of Chapter - 4)**

(ii) $x(t) = e^{-2t} u(t+2)$ **(Refer Q.17 (ii) of Chapter - 4)** **[6]**

b) *Find and sketch exponential Fourier series of the given signal :*
(Refer Q.13 of Chapter - 3) **[6]**

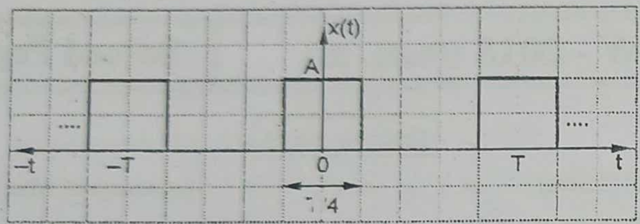


Fig. 1

OR

Q.4 a) *Find and sketch the trigonometric Fourier series of train of impulse defined as :* $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$. **(Refer Q.17 of Chapter - 3)** **[6]**

b) *Find Fourier transform of the following signals :* **[6]**

i) $u(t)$

ii) $sgn(t)$ **(Refer Q.11 of Chapter - 4)**

Q.5 a) *Find the Laplace transform of the following :*

(i) $x(t) = \frac{d}{dt} t e^{-t} u(t)$ **[3]**

(ii) $x(t) = e^{-3t} u(t) * \cos(t-2) u(t-2)$ **[4]**

Ans. : i) $e^{-t} u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+1}$

By differentiation in s - domain, $t e^{-t} u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+1)^2}$

By differentiation in time domain, $\frac{d}{dt}[t e^{-t} u(t)] \xleftrightarrow{\mathcal{L}} \frac{s}{(s+1)^2}$

ii) Here $e^{-3t} u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+3}$ and $\cos t \cdot u(t) \xleftrightarrow{\mathcal{L}} \frac{s}{s^2+1}$

By time shifting property, $\cos(t-2) u(t-2) \xleftrightarrow{\mathcal{L}} e^{-2s} \cdot \frac{s}{s^2+1}$

By convolution in time,

$$e^{-3t} u(t) * \cos(t-2) u(t-2) \xleftrightarrow{\mathcal{L}} \frac{e^{-2s} \cdot s}{(s+3)(s^2+1)}.$$

b) *Find initial and final values of the signal x(t) having unilateral Laplace transform :* **[6]**

(i) $X(s) = \frac{7s+10}{s(s+2)}$ **(Refer Q.23 of Chapter - 5)**

(ii) $X(s) = \frac{5s+4}{s^3+3s^2+2s}$ **(Refer similar Q.26 of Chapter - 5)**

OR

Q.6 a) *Find inverse Laplace transform of :*

$$X(s) = \frac{3s+7}{(s^2-2s-3)}$$

for :

(i) $s > 3$

(ii) $s < -1$

(iii) $-1 < s < 3$ **(Refer Q.31 of Chapter - 5)**

b) *Find transfer function and impulse response of the causal system described by the differential equation :*

$$\frac{d^2}{dt^2} y(t) + 5\frac{d}{dt} y(t) + 6y(t) = 2\frac{d}{dt} x(t) - 3x(t)$$

(Refer Q.36 of Chapter - 5)

Q.7 a) *Find auto-correlation function of the signal given, using graphical method :*

$x(n) = \{2, 1, -2, 1, 3\}$

Ans. : Here $x(n) = \{2, 1, \underset{\uparrow}{-2}, 1, 3\}$, Hence $x(-n) = \{3, 1, \underset{\uparrow}{-2}, 1, 2\}$ [6]

Autocorrelation is given as, $R_{xx}(m) = x(m) * x(-m)$

$x(m) \Rightarrow$				2	1	$\underset{\uparrow}{-2}$	1	3	
$x(-m) \Rightarrow$				3	1	$\underset{\uparrow}{-2}$	1	2	
					4	2	−4	2	6
				2	1	−2	1	3	×
			−4	−2	4	−2	−6	×	×
		2	1	−2	1	3	×	×	×
	6	3	−6	3	9	×	×	×	×
	6	5	−9	1	$\underset{\uparrow}{19}$	1	−9	5	6

$\therefore \quad R_{xx}(m) = \{6, 5, -9, 1, \underset{\uparrow}{19}, 1, -9, 5, 6\}.$

b) *The probability density function of a random variable X is given by :*

$f_X(x) = e^{-x} u(x)$

determine :

(i) CDF *(ii) P (X ≤ 1)*

(iii) P (1 < X ≤ 2) *(iv) P (X > 2)*

(Refer Q.16 of Chapter - 6) [7]

OR

Q.8 a) *The probability density function of a random variable 'X' is given by :*

$$f_X(x) = \begin{cases} \dfrac{1}{a} & |x| \le a \\ 0 & \text{otherwise} \end{cases}$$

determine :

(i) Mean E [X]

(ii) Mean square value $E[X^2]$

(iii) Standard deviation **(Refer Q.23 of Chapter - 6)** [7]

b) *State and prove the relationship between auto-correlation and energy spectral density of energy signal.* [6]

Ans. : Consider the definition of autocorrelation for periodic functions :

$$R(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_p(t)\, x_p^*(t-\tau)\, dt \quad \dots (1)$$

Here $x_p(t)$ represents periodic signal. We know that a periodic power signal $x_p(t)$ can be represented as,

$$x_p(t) = \sum_{m=-\infty}^{\infty} x(t - mT_0) \quad \dots (2)$$

With the above equation we can write equation (1) as,

$$R(\tau) = \frac{1}{T_0} \int_{-\infty}^{\infty} x(t) \sum_{m=-\infty}^{\infty} x^*(t - \tau - mT_0)\, dt$$

The above equation uses limits $(-\infty, \infty)$ since $x(t)$ is non-periodic signal. Let us rearrange the above equation as,

$$R(\tau) = \frac{1}{T_0} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} x(t)\, x^*[t - (\tau + mT_0)]\, dt \quad \dots (3)$$

The autocorrelation function of energy signals is given as,

$$R'(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt$$

Here we have used $R'(\tau)$ notation to indicate that it is different from $R(\tau)$. Applying the above definition to equation (3) we have,

$$R(\tau) = \frac{1}{T_0} \sum_{m=-\infty}^{\infty} R'(\tau + mT_0) \quad \dots (4)$$

Here $R(\tau)$ = Autocorrelation function of periodic signals and

$R'(\tau)$ = Autocorrelation function of energy signals.

Take Fourier transforms of both sides of above equation,

$$F[R(\tau)] = \frac{1}{T_0} \sum_{m=-\infty}^{\infty} F[R'(\tau + mT_0)]$$

Since $F[R'(\tau)] = \psi(f)$ and using time shifting property of Fourier transform,

$$F[R(\tau)] = \frac{1}{T_0} \sum_{m=-\infty}^{\infty} \psi(f)\, e^{jm\omega T_0}$$

$$= \frac{1}{T_0} \sum_{m=-\infty}^{\infty} \psi(f)\, e^{j2\pi f m T_0}, \text{ since } \omega = 2\pi f$$

since $\psi(f) = |X(f)|^2$ the above equation will be,

$$F[R(\tau)] = \frac{1}{T_0} |X(f)|^2 \sum_{m=-\infty}^{\infty} e^{j2\pi m f T_0} \quad \text{... (5)}$$

Now $\sum_{m=-\infty}^{\infty} e^{j2\pi m f T_0}$ can be written as delta function placed at multiples of f_0 i.e.,

$$\sum_{m=-\infty}^{\infty} e^{j2\pi m f T_0} = \frac{1}{T_0} \sum_{k=-\infty}^{\infty} \delta(f - k f_0) \quad \text{... (6)}$$

Therefore equation (5) will be,

$$F[R(\tau)] = \frac{1}{T_0} |X(f)|^2 \cdot \frac{1}{T_0} \sum_{k=-\infty}^{\infty} \delta(f - k f_0)$$

$$= \frac{1}{T_0^2} \sum_{k=-\infty}^{\infty} |X(k f_0)|^2\, \delta(f - k f_0)$$

$$= S(f) \qquad \text{since } f = k f_0 \text{ for periodic signals.}$$

Thus, $R(\tau) \leftrightarrow S(f)$ is the required relation.

This pair is written as,

$$S(f) = \int_{-\infty}^{\infty} R(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

$$\text{and } R(\tau) = \int_{-\infty}^{\infty} S(f)\, e^{j2\pi f\tau}\, df \quad \text{... (7)}$$

DECEMBER - 2018 [5459] - 131 Solved Paper

Course 2015

Time : 2 Hours] [Maximum Marks : 50

Q.1 a) *Perform the following operations on the given signal x(t) which is defined as :*

$x(t) = u(t + 4)$

1) Sketch $z(t) = x(-t-1)$

2) Sketch $y(t) = x(t) + z(t)$. **(Refer Q.13 of Chapter - 1)** [4]

b) *Write the expression for energy and power of the signal. Also determine whether the following signals is energy or power and find energy or time averaged power of the signal :*

$x(t) = 5\cos(10\pi t) + \sin(20\pi t); \quad -\infty \le t \le \infty.$ [4]

(Refer Q.25 of Chapter - 1)

c) *Determine whether the following system is static/dynamic, casual/no-casual and stable/unstable and justify :* $h(t) = e^{-10t} u(t)$.

(Refer Q.21 of Chapter - 2) [5]

OR

Q.2 a) *Compute the convolution integral by graphical method and sketch the output for the following signals :*

$x(t) = u(t)$

$h(t) = e^{-2t} u(t)$. **(Refer Q.6 of Chapter - 2)** [5]

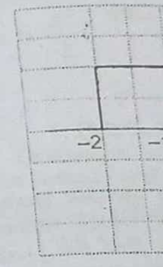
b) *Check whether the following signal is even or odd and determine the even and odd part of the signal :*

$x(t) = u(t)$ **(Refer Q.19 (i) of Chapter - 1)** [4]

c) *Compute the convolution integral for the following signal :*

$x(t) = u(t),\ h(t) = \delta(t+1) + \delta(t) + \delta(t-1)$ **(Refer Q.7 of Chapter - 2)** [4]

.3 a) *Find the trignome*
hown in the following figur



b) *State any six*
Refer Q.9 of Chapter

Ans. : Duality :

$X(t$

Proof :

Inverse Fourier T

$x(t)$

Interchanging t by ω

$x(\omega$

Interchanging ω by

$x(-$

i.e. $2\pi x(-$

Q.3 a) *Find the trignometric Fourier series for the periodic signal x(t) shown in the following figure :* (Refer Q.11 of Chapter - 3) [6]

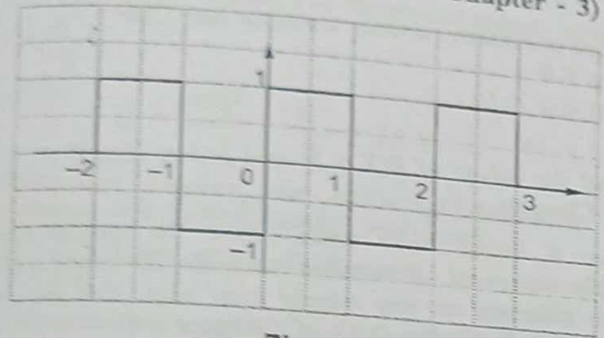


Fig. 1

b) *State any six properties of Fourier transform.*
(Refer Q.9 of Chapter - 4)

Ans. : **Duality :** [6]

$$X(t) \xleftrightarrow{FT} 2\pi\, x(-\omega)$$

Proof :

Inverse Fourier Transform is given as,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

Interchanging t by ω we get,

$$x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(t)\; e^{j\omega t}\, dt$$

Interchanging ω by $-\omega$ we get,

$$x(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(t)\; e^{-j\omega t}\, dt$$

$$2\pi\, x(-\omega) = \int_{-\infty}^{\infty} X(t)\; e^{-j\omega t}\, dt$$

Right hand side of above equation is Fourier transform of $X(t)$ i.e.,

$$X(t) \xleftrightarrow{FT} 2\pi x(-\omega)$$

Parseval's Theorem or Rayleigh's Theorem :

Refer Q.10 of Chapter - 4.

OR

Q.4 a) *Find the Fourier transform of the following signals :* [6]
1) $x(t) = sgn\ (t)$ (Refer Q.11 (ii) of Chapter - 4)
2) $x(t) = cos\ (\omega_0\ t)\ u(t)$ (Refer Q.5 (ii) of Chapter - 4)

b) *Write expression for trignometric fourier series and exponential fourier series.* (Refer Q.5 of Chapter - 3) [4]

c) *Define amplitude and phase spectra of the signal.* [2]

Ans. : Amplitude spectra of the signal is given by,

$$|C(k)| = \sqrt{a^2(k) + b^2(k)}$$

Phase spectra of the signal is given by,

$$\angle C(k) = \tan^{-1}\frac{b(k)}{a(k)}$$

Q.5 a) *Find the inverse Laplace transform of*
$X(s) = \dfrac{2}{(s+4)(s-1)}$
If the Region of convergence is : (Refer Q.32 of Chapter - 5) [6]
1) $-4 \le \text{Re}(s) \le 1$
2) $\text{Re}(s) > 1$
3) $\text{Re}(s) < -4.$

b) *A signal x(t) has Laplace transform :* [6]
$X(s) = \dfrac{s+2}{s^2+4s+5}$
Find the Laplace transform of the following signals
1) $y_1(t) = t\,x(t)$
2) $y_2(t) = e^{-t}x(t)$ (Refer Q.13 of Chapter - 5)

OR

Q.6 a) *Find the Laplace transform of the following signal and sketch ROC :*

$x(t) = e^{-3t}u(t) + e^{-5t}u(t)$ **(Refer Q.14 of Chapter - 5)**

b) *Find the initial and final value of the following signal :* **[6]**

$X(s) = \dfrac{2s+3}{s^2+5s-7}$ **(Refer Q.25 of Chapter - 5)** **[4]**

c) *State the relationship between Fouier transform and Laplace transform.* **(Refer Q.1 of Chapter - 5)**

Q.7 a) *Find the following for the given signal x(t) :* **[2]**

i) Autocorrelation

ii) Energy from Autocorrelation **[6]**

$x(t) = e^{-10t}u(t)$.

Ans. : Given : $x(t) = e^{-10t} \cdot u(t)$

1) Autocorrelation :

$$R(\tau) = \int_{-\infty}^{\infty} x(t).x(t-\tau)\,dt$$

$$= \int_{-\infty}^{\infty} e^{-10t} \cdot u(t).e^{-10(t-\tau)} \cdot u(t-\tau)\,dt$$

$$= e^{10\tau} \int_{\tau}^{\infty} e^{-20t}\,dt$$

$$\boxed{R(\tau) = \frac{e^{-10\tau}}{20}}$$

2)

$$\boxed{\text{Energy} = R(0) = \frac{1}{20}}$$

b) *Define probability and state the properties of PDF. Also state the relationship between CDF and PDF.*

(Refer Important Points to Remember of section 6.1 and Q.11 of Chapter - 6) **[7]**

OR

Q.8 a) *Suppose a certain random variable has CDF :*

$F_x(x) = 0, \quad x \le 0$

$F_x(x) = kx^2, \quad 0 < x \le 10$

$F_x(x) = 100k, \quad x > 10$

Calculate k. Find the values of $P(X \le 5)$ *and* $P(5 < X \le 7)$.

(Refer Q.15 of Chapter - 6) **[7]**

b) *A coin is tossed three times. Write the sample space which gives all possible outcomes. A random variable X, which reprsents the number of heads obtained on any tripple toss. Also, find the probabilties of X and plot the C.D.F.* **(Refer Q.13 of Chapter - 6)** **[6]**

MAY - 2019 [5559] - 131 Solved Paper

Course 2015

Time : 2 Hours] **[Maximum Marks : 50**

Q.1 a) *Perform the following operations on the given signal x(t) which is defined as :*

$x(t) = 2 * rect\left(\frac{t}{4}\right)$

Sketch $z(t) = x(-t-1)$ **(Refer Q.14 of Chapter - 1)** **[2]**

b) *Write the expression for energy and power of the signal. Also determine whether the following signal is energy or power and find energy or time averaged power of the signal :*

$x(t) = 5\cos(10\pi t) + 5\sin(20\pi t); \quad -\infty \le t \le \infty$. **[4]**

Ans. : Energy can be expressed as,

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \cdot dt$$

Power cab be expressed as,

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \cdot dt$$

Given signal is,

$x(t) =$

∴ Given signal is a power component is $\frac{A^2}{2}$

∴ P =

c) Determine w Causal/Non causal and $h(t) = 2^* rect\left(\frac{t}{10}\right)$ (Re

d) Determine t impulse response is h((Refer Similar Q.24

Q.2 a) Compute t sketch the output for $x(t) = u(t)$ $h(t) = e^{-}$

b) Check w determine the even (Refer Q.19 of Ch

c) Compute $x(t) = u(t), h(t) =$

d) Determin periodic find the f $x(t) = \cos(t$

Q.3 a) Find t periodic signal x((Refer Q.11 of C

Given signal is,

$$x(t) = 5\cdot\cos(10\pi t) + 5\cdot\sin(20\pi t) \; ; \; -\infty \le t \le \infty$$

$\therefore$ Given signal is a power signal. For sinusoidal signal, power of ndividual component is $\frac{A^2}{2}$.

$$\therefore \quad P = \frac{(5)^2}{2} + \frac{(5)^2}{2} = 25 \text{ W}$$

c) *Determine whether the following system is static/dynamic. Causal/Non causal and stable/Unstable and justify.*
$h(t) = 2 * rect\left(\frac{t}{10}\right)$ **(Refer Q.22 of Chapter - 2)** [3]

d) *Determine the step response of the following systems whose impulse response is* $h(t) = e^{-5t}u(t)$.
(Refer Similar Q.24 of Chapter - 2) [4]

OR

Q.2 a) *Compute the convolution integral by graphical method and sketch the output for the following signals :*

$x(t) = u(t)$

$h(t) = e^{-2t}u(t)$. **(Refer Q.6 of Chapter - 2)** [4]

b) *Check whether the following signal is even or odd and determine the even and odd part of the signal :* $x(t) = u(t)$.
(Refer Q.19 of Chapter - 1) [4]

c) *Compute the convolution integral for the following signal :*
$x(t) = u(t),\ h(t) = \delta(t+1) + \delta(t) + \delta(t-1)$. **(Refer Q.7 of Chapter - 2)** [2]

d) *Determine whether the following signals are periodic or not, if periodic find the fundamental period of the signal*

$x(t) = \cos(t) + \sin(2t)$ **(Refer Q.29 of Chapter - 1)**

Q.3 a) *Find the trigonometric/exponential Fourier series for the periodic signal x(t) shown in the following figure :*
(Refer Q.11 of Chapter - 3) [6]

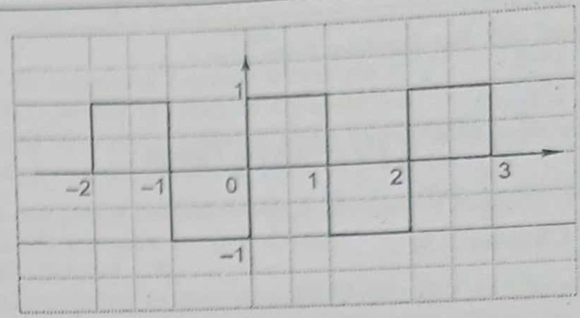


Fig. 1

b) *Find Fourier transform of the following signal* [6]

$\frac{d}{dt}\{(e^{-3t}u(t) * e^{-3t}u(t-2))\}$ **(Refer Q.18 of Chapter - 4)**

OR

Q.4 a) *Find the Fourier transform of the following signals :* [4]

1) $x(t) = \sin(t)$

2) $x(t) = \cos(\omega_0 t)u(t)$ **(Refer Q.5 (ii) of Chapter - 4)**

Ans. : 1) Given : $x(t) = \sin t$

$$x(\omega) = \frac{\pi}{j}[\delta(\omega-1) - \delta(\omega+1)]$$

b) *State the dirichlet condition for existence of Fourier series. Define amplitude and phase-spectrum.* **(Refer Q.1 of Chapter - 3)** [5]

Ans. : Magnitude spectrum - The plot of magnitude with respect to frequency is called mangitude spectrum. It is given as,

$$|c(k)| = \sqrt{a^2(k) + b^2(k)} \quad \text{or} \quad |X(k)|$$

Phase spectrum - The plot of phase shift with respect to frequency is called phase spectrum. It is given as,

$$\angle c(k) = \tan^{-1}\frac{b(k)}{a(k)} \quad \text{or} \quad \angle X(k)$$

c) *Write expression for trigonometric Fourier series and exponential Fourier series.* **(Refer Q.5 of Chapter - 3)** [3]

Q.5 a) *Find the inverse Laplace transform of* $X(s) = \frac{2}{(s+4)(s-1)}$

If the Region of convergence is :

1) $-4 \le Re(s) < 1$ *2)* $Re(s) > 1$ *3)* $Re(s) < -4$.

(Refer Q.32 of Chapter - 5) [6]

b) A signal x(t) has Laplace transform :

$$X(s) = \frac{s+2}{s^2+4s+5}$$

Find the Laplace transform of the following signals.

a) $y_1(t) = \frac{d}{dt}(x(t))$ b) $y_2(t) = x(2t)$ [6]

(Refer Q.19 of Chapter - 5)

OR

Q.6 a) *Find the Laplace transform of the following signal and sketch ROC :* $x(t) = e^{-3t}u(t) + e^{-5t}u(t)$. **(Refer Q.14 of Chapter - 5)**

b) *Find the initial and final value of the following signal :* [6]

$$X(s) = \frac{2s+3}{s^2+5s-7}$$ **(Refer Q.25 of Chapter - 5)** [4]

c) *State the relationship between Fourier transform and Laplace transform.* **(Refer Q.1 of Chapter - 5)**

Q.7 a) *Find the following terms* [2]

1) Autocorrelation 2) Cross correlation **(Not in New Syllabus)** [2]

b) *State the properties of Probability Density Function (PDF).* **(Refer Q.11 of Chapter - 6)** [3]

c) *A random variable X has PDF.* [4]

$$f_X(x) = \begin{cases} 5x^2 & ; \quad 0 \le x < 1 \\ 0 & ; \quad elsewhere \end{cases}$$

Find E[X], E[3X – 2], $E[X^2]$ and standard deviation.

(Refer Q.21 of Chapter - 6)

d) *Explain uniform distribution model with respect to its density and distribution function.* **(Refer Q.6 of Chapter - 6)** [4]

OR

Q.8 a) *Consider the experiment as rolling of two dice. Find the CDF for the random variable X if it assigns the sum of numbers appearing on the dice to each outcome.* **(Refer Q.14 of Chapter - 6)** [6]

b) *A box contains 3 white, 4 red and 5 black balls. A ball is drawn at random find the probability that it is 1) Red 2) Not black 3) Black or white.* [3]

Ans. : Refer Q.2 of Chapter - 6

iii) *P*(**black or white**)

Here $N(\text{black}) = {}^5C_1 = 5$

and $N(\text{white}) = {}^3C_1 = 3$

$\therefore P(\text{black or white}) = P(\text{black}) + P(\text{white})$

$$= \frac{N(\text{black})}{N} + \frac{N(\text{white})}{N} = \frac{5}{12} + \frac{3}{12} = \frac{8}{12} = \frac{2}{3}$$

c) *Explain Gaussian distribution model with respect to its density and distribution function.* **(Refer Q.7 of Chapter - 6)** [4]

DECEMBER - 2019 [5668] - 131 | Solved Paper

Course 2015

Time : 2 Hours] **[Maximum Marks : 50**

Q.1 a) *Check whether the following systems are Causal, Time variant and Linear and Justify.*

1) $x^2(t) + x(t+2)$ *2)* $Ax(n) + B$. **(Refer Q.36 of Chapter - 1)** [6]

b) *Sketch the waveforms for the following signals :*

1) $x(t) = u(t+1) - 2u(t) + u(t-1)$;

2) $x(t) = \sum_{k=-\infty}^{k=\infty} \delta(t-2k)$. **(Refer Q.15 of Chapter - 1)** [4]

c) *Check whether the following system is stable / unstable, causal / non-causal and static / dynamic whose impulse response is :*

$h(t) = e^{-2t}u(t)$

Also justify the same. **(Refer Q.23 of Chapter - 2)** [3]

OR

Q.2 a) *Find the step response of systems whose impulse responses are given by :*

1) $h(t) = u(t+1) - u(t-1)$

2) $h(t) = \delta(t) - \delta(t-1)$. **(Refer Q.24 of Chapter - 2)** [4]

b) *Compare the convolution integral by graphical method and sketch the output for :* $x_1(t) = u(t-2)$ $h(t) = u(t)$.

(Refer Q.4 of Chapter - 2) [5]

c) Determine whether the following signal is periodic or not, if periodic find the fundamental period of the signal : 1) $x(t) = \cos^2(2\pi t)$ 2) $x(t) = e^{-2t}u(t)$. **(Refer Q.30 of Chapter - 1)** [4]

Q.3 a) Find the trigonometric exponential Fourier series for the periodic signal x (t). Sketch the amplitude and phase spectra. **(Refer Q.11 of Chapter - 3)** [6]

b) Find the Fourier transform of $x(t) = rect\left(\frac{t}{\tau}\right)$ and sketch the magnitude and phase spectrum. **(Refer Q.14 of Chapter - 4)** [6]

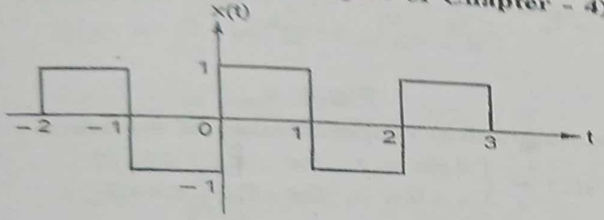


Fig.1

OR

Q.4 a) State the dirichlet conditions for existence of Fourier transform. **(Refer Q.2 of Chapter - 4)** [3]

b) Find the Fourier transform of :

(1) $x(t) = \cos(\omega_0 t)$ **(Refer Q.5 of Chapter - 4)**

(2) $x(t) = e^{-2t}u(t)$. **(Refer Q.3(i) of Chapter - 4)** [6]

c) Explain Gibb phenomenon. **(Refer Q.9 of Chapter - 3)** [3]

Q.5 a) If $X(x) = \frac{2}{(s+3)}$ find Laplace transform of

(1) $\frac{d}{dt}(x(t))$ (2) $t\,x(t)$. **(Refer Q.20 of Chapter - 5)** [6]

b) Find inverse Laplace transform of $X(s) = \frac{3s^2+8s+6}{(s+2)(s^2+2s+1)}$.

(Refer Q.33 of Chapter - 5) [6]

OR

Q.6 a) Find the unilateral Laplace transform of :

(1) $\delta(t)$ **(Refer Q.10(ii) of Chapter - 5)**

(2) $x(t) = \cos(\omega_0 t)$ **(Refer Q 28 of Chapter - 5)**

(3) $x(t) = u(t)$. **(Refer Q.10(i) of Chapter - 5)** [6]

b) State and prove the following properties of Laplace transform :

(1) Differentiation in time domain **(Refer Q.6(i) of Chapter - 5)**

(2) Convolution in time domain **(Refer Q.7 of Chapter - 5)**

(3) Time shifting. **(Refer Q.5 of Chapter - 5)** [6]

Q.7 a) State any three properties of autocorrelation signals. **(Not in New Syllabus)** [3]

b) Explain Gaussian probability model with respect to its density and distribution function. **(Refer Q.7 of Chapter - 6)** [4]

c) Find the mean, second moment and standard deviation of X, if pdf, $f_X(X) = e^{-3x}u(x)$. **(Refer Q.22 of Chapter - 6)** [3]

d) A box contains 10 white, 15 red and 15 black balls. A ball is drawn at random find the probability that it is : (1) Red (2) Not black (3) Black or white. **(Refer Q.5 of Chapter - 6)** [3]

OR

Q.8 a) A coin is tossed three times. Write the sample space which gives all possible outcomes. A random variable X, which represents the number of heads obtained on any triple toss. Also find the probabilities of X and plot the C.D.F. **(Refer Q.13 of Chapter - 6)** [7]

b) Suppose a certain random variable has CDF :

$F_x(x) = 0,\ x \le 0$

$F_x(x) = kx^2,\ 0 < x \le 10$

$F_x(x) = 100\,k,\ x > 10$

Calculate K. Find the values of $P(X \le 5)$ and $P(5 < X \le 7)$.

(Refer Q.15 of Chapter - 6) [6]

JUNE - 2022 [5869] - 246 — Solved Paper

Course 2019

Time : 2½ Hours] [Maximum Marks : 70

Instructions to the candidates :

1) *Neat diagrams must be drawn wherever necessary.*

2) *Figures to the right indicate full marks.*

3) *Use of logarithmic tables, slide-rule, mollier charts, electronic pocket calculator steam tables is allowed.*

4) *Assume suitable data, if necessary.*

Q.1 a) *What is fourier series. What are the methods of finding fourier series. Write their expressions.* **(Refer Q.5 of Chapter - 3)** **[6]**

b) *State the following properties of DT fourier series.* **[6]**

i) Time scaling ii) Linearity iii) Convolution

(Refer Q.22 and Q.18 of Chapter - 3)

c) *Find out the exponential fourier series for impulse train shown in Fig.1 below. Also plot it's magnitude and phase spectrum.*
(Refer Q.14 of Chapter - 3) **[6]**

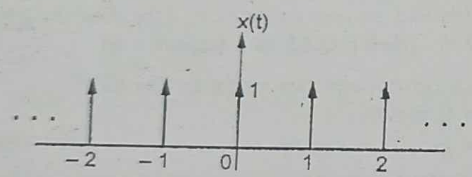


Fig.1

OR

Q.2 a) *Explain Gibb's phenomenon for fourier series.*
(Refer Q.9 of Chapter - 3) **[4]**

b) *Determine the fourier series for the signal with the per wave as shown in Fig.2 below.*

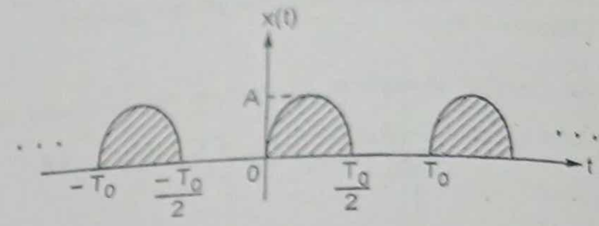


Fig.2

Ans. : Step 1 : Mathematical representation of waveform

$$x(t) = \begin{cases} A\sin\omega_0 t & \text{for } 0 \le t \le T_0/2 \\ 0 & \text{for } T_0/2 \le t < T_0 \end{cases}$$

And $T_0 = T = 2\pi$. Therefore $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$

Step 2 : To obtain Fourier coefficients

$$X(k) = \frac{1}{T}\int_{<T>} x(t)e^{-jk\omega_0 t}\,dt \text{ By definition}$$

$$= \frac{1}{2\pi}\int_0^{\pi} A\sin\omega_0 t\, e^{-jk\omega_0 t}\,dt$$

$$= \frac{A}{2\pi}\int_0^{\pi} \sin t\, e^{-jkt}\,dt \text{ since } \omega_0 = 1 \quad ...($$

Here let us use, $\int e^{ax}\sin(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin(bx+c)-b\cos(bx+$

with $a = -jk$, $b = 1$, $c = 0$ and $x = t$. Then above integration will be,

$$X(k) = \frac{A}{2\pi}\left\{\frac{e^{-jkt}}{(-jk)^2+1}[-jk\sin t - \cos t]\right\}_0^{\pi}$$

$$= \frac{A}{2\pi(-jk)^2+1}\{e^{-j\pi k}+1\}$$

Here $(-jk)^2 = (-j)^2 k^2 = -k^2$ and $e^{-j\pi k} = (-1)^k$ i.e.,

$$X(k) = \frac{A}{2\pi(1-k^2)}[(-1)^k + 1] \text{ for } k \neq \pm 1$$

$$= \begin{cases} \dfrac{A}{\pi(1-k^2)} & \text{for } k = 0, \pm 2, \pm 4, \pm 6, \ldots \\ 0 & \text{for } k = \pm 3, \pm 5, \ldots \end{cases}$$

Evaluating equation (1) for $k = 1$ and -1 separately,

$$X(k) = \frac{A}{2\pi}\int_0^{\pi} \sin t\, e^{-jt}\, dt \text{ for } k = 1$$

$$= \frac{A}{2\pi}\int_0^{\pi} \frac{e^{jt} - e^{-jt}}{2j} \cdot e^{-jt}\, dt = \frac{A}{j4}$$

Similarly, $$X(k) = \frac{A}{2\pi}\int_0^{\pi} \sin t\, e^{jt}\, dt \text{ for } k = -1$$

$$= \frac{A}{-j4}$$

Magnitude and phase of $X(k)$ can be written as,

$$|X(k)| = \begin{cases} \left|\dfrac{A}{\pi(1-k^2)}\right| & \text{for } k = 0, \pm 2, \pm 4, \pm 6, \ldots \\ \dfrac{A}{4} & \text{for } k = \pm 1 \\ 0 & \text{for } k = \pm 3, \pm 5, \ldots \end{cases}$$

Fig. 3 shows the magnitude and phase plots.

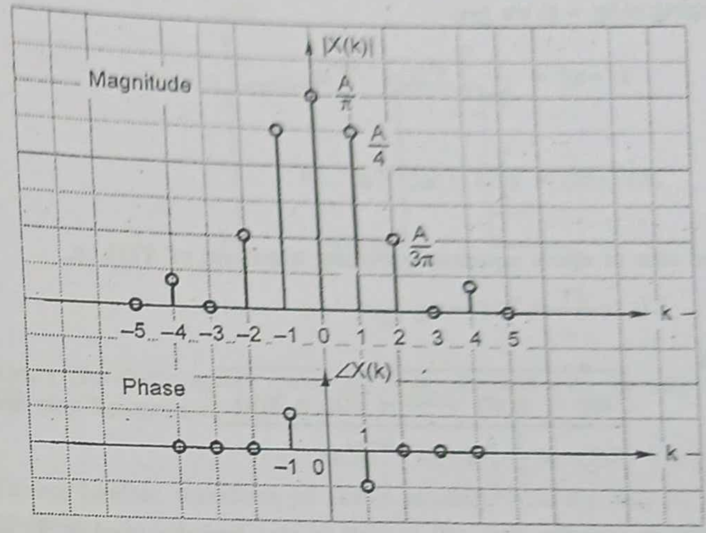


Fig. 3 Magnitude and phase plots of half wave rectified sine wave

c) *State the following properties of fourier series.* [6]

i) Duality ii) Time bandwidth iii) Parseval's relation

(Refer Q.8 of Chapter - 3)

Ans. : i) Duality :

$$X(t) \xleftrightarrow{FT} 2\pi x(-\omega) \quad \ldots(1)$$

Proof : Inverse Fourier transform is given as,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega$$

Interchanging t by ω we get,

$$x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(t) e^{j\omega t}\, dt$$

Interchanging ω by − ω we get,

$$x(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(t)e^{-j\omega t}\,dt$$

i.e. $$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(t)e^{-j\omega t}\,dt$$

Right hand side of above equation is Fourier transform of $X(t)$ i.e.,

$$X(t) \xleftrightarrow{FT} 2\pi x(-\omega)$$

ii) Scaling

$$\boxed{z(t) = x(at) \xleftrightarrow{FS} Z(k) = X(k)} \quad \ldots (2)$$

Proof : $$X(k) = \frac{1}{T}\int_{<T>} x(t)\, e^{-jk\omega_0 t}\,dt$$

- Since x(t) is periodic, then z(t) = x(at) is also periodic. And if 'T' is the period of x(t), then period of z(t) will be $\frac{T}{a}$.
- Similarly if frequency of x(t) is ω_0, the frequency of z(t) = x(at) will be $a\omega_0$, since 't' is multiplied by factor 'a'.

Therefore Fourier coefficients of z(t) can be written as,

$$Z(k) = \frac{1}{\left(\frac{T}{a}\right)}\int_{\left(\frac{T}{a}\right)} z(t)e^{-jk(a\omega_0)t}\,dt = \frac{a}{T}\int_{\left(\frac{T}{a}\right)} x(at)\, e^{-jka\omega_0 t}\,dt$$

Put at = m, then dt = $\frac{1}{a}$ dm, then above equation becomes,

$$Z(k) = \frac{a}{T}\int_{\left(\frac{T}{a}\right)} x(m)e^{-jk\omega_0 m}\cdot\frac{1}{a}\,dm$$

$$= \frac{1}{T}\int_{\left(\frac{T}{a}\right)} x(m)\, e^{-jk\omega_0 t}\,dm = X(k)$$

Comment : Fourier coefficients of x(t) and x(at) are same, but spacing between frequency components change from ω_0 to $a\,\omega_0$.

Q.3 a) *Find the Fourier transform of the signal $x(t) = e^{-at}u(t)$ sketch magnitude and phase response.* **(Refer Q.3 of Chapter - 4)**

b) *Obtain the Fourier transform using the property.*

i) $x(t) = \frac{d}{dt}[e^{-at}\,u(t)]$ **(Refer Q.17 (i) of Chapter - 4)**

ii) $x(t) = \delta(t) + e^{-at}u(t)$

Ans. : ii) x(t) $= \delta(t) + e^{-at}\,u(t)$

We have standard fourier transform pairs,

$$\delta(t) \xleftrightarrow{FT} 1$$

$$e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{a+j\omega}$$

Using linearly property of fourier transform

$$FT\{\delta(t) + e^{-at}u(t)\} = FT\{\delta(t)\} + FT\{e^{-at}u(t)\}$$

$$= 1 + \frac{1}{a+j\omega}$$

c) *State and explain Dirichelet's conditions for the existence of Fourier transform.* **(Refer Q.2 of Chapter - 4)**

OR

Q.4 a) *State any six properties of Fourier transform.* **(Refer Q.8, Q.9 and Q.10 of Chapter - 4)**

b) *Obtain Inverse Fourier transform of the signal given below.* **(Refer Q.14 of Chapter - 4)**

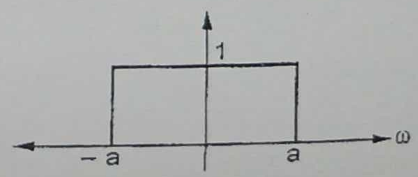


Fig. 4

c) *Define magnitude response and phase response. Obtain the Fourier transform of impulse response. Also sketch magnitude response of impulse signal.* **(Refer Q.4 of Chapter - 4)** [5]

Q.5 a) *Find the initial and final value of the given function.* [6]

$$X(s) = \frac{s+2}{s^2+5s+7}$$ **(Refer Q.27 of Chapter - 5)**

b) *State the limitations of Fourier transform and need of Laplace transform. Compare both.* **(Refer Q.3 of Chapter - 5)** [6]

c) *Given the Laplace transform of* $X(s) = \frac{2s}{s^2+2}$.

Determine x(t) and Laplace transform x(3t) and x(t–2). [6]

Ans. : Using time scaling property,

$$x(at) \xleftrightarrow{\mathcal{L}} \frac{1}{|a|} X\left(\frac{s}{a}\right)$$

$$x(3t) \xleftrightarrow{\mathcal{L}} \frac{1}{3} X\left(\frac{s}{3}\right) = \frac{2\left(\frac{s}{3}\right)}{\left(\frac{s}{3}\right)^2+2} = \frac{6s}{s^2+18}$$

Using the time shift property,

$$x(t-t_0) \xleftrightarrow{\mathcal{L}} e^{-st_0} X(s)$$

$$\therefore x(t-2) \xleftrightarrow{\mathcal{L}} e^{-2s} \cdot \frac{2s}{s^2+2}$$

OR

Q.6 a) *State any six properties of Laplace transform.* **(Refer Q.5 to Q.8 of Chapter - 5)** [6]

b) *Find the Laplace transform of periodic wave given below.* [6]

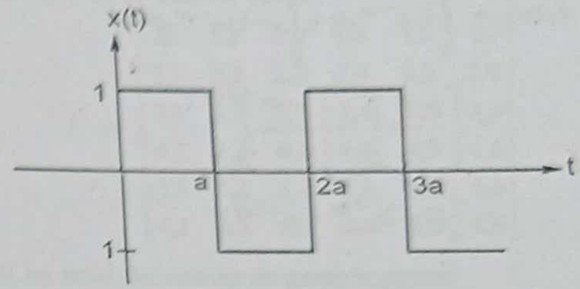


Fig. 5

Ans. : The above square wave can be represented with the help of shifted unit step functions as follows :

$$x(t) = u(t) - 2u(t-a) + 2u(t-2a) - 2u(t-3a) + \ldots$$

Taking Laplace transform of above equation with the help of time shift property.

$$\mathcal{L}[x(t)] = \frac{1}{s} - 2\frac{1}{s}e^{-as} + 2\frac{1}{s}e^{-2as} - 2\frac{1}{s}e^{-3as} + \ldots$$

$$= \frac{1}{s}[1 - 2e^{-as} + 2e^{-2as} - 2e^{-3as} + \ldots]$$

$$= \frac{1}{s}\{1 - 2e^{-as}[1 - e^{-as} + e^{-2as} - e^{-3as} + \ldots]\}$$

$$= \frac{1}{s}\left\{1 - 2e^{-as} \cdot \frac{1}{1+e^{-as}}\right\}$$

$$= \frac{1}{s} \cdot \frac{e^{as}-1}{e^{as}+1} = \frac{1}{s}\tanh\left(\frac{as}{2}\right) \quad \ldots(1)$$

Second method : The first cycle of square wave of Fig. 6 can be represented as,

$$x_1(t) = \begin{cases} 1 & \text{for } 0 \le t \le t\ a \\ -1 & \text{for } a \le t \le 2a \end{cases}$$

$$\therefore \quad \mathcal{L}\, x_1(t) = \int_0^a 1\cdot e^{-st}dt + \int_0^a (-1)e^{-st}dt$$

$$= \left[\frac{e^{-st}}{s}\right]_0^a + \left[\frac{e^{-st}}{s}\right]_0^{2a}$$

$$= \frac{1}{s}[1-2e^{-as}+e^{-2as}]$$

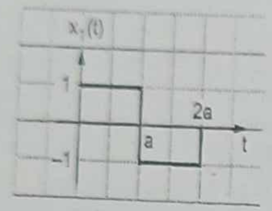


Fig. 6

The square wave of Fig. 6 repeats with period $T = 2a$. The Laplace transform of a periodic function is given as,

$$X(s) = \frac{1}{1-e^{-sT}}X_1(s) = \frac{1}{1-e^{-2as}}\cdot\frac{1}{s}[1-2e^{-as}+e^{-2as}]$$

$$= \frac{(1-e^{-as})^2}{s(1-e^{-as})(1+e^{-as})}$$

$$= \frac{1}{s}\frac{1-e^{-as}}{1+e^{-as}} = \frac{1}{s}\frac{e^{as}-1}{e^{as}+1} = \frac{1}{s}\tanh\left(\frac{as}{2}\right)$$

c) *Find the Inverse Laplace transform of* $X(s) = \dfrac{2}{s(s+1)(s+2)}$ *with ROC specified as* $-1 < Re\ CS >< 0$. **(Refer Q.30 of Chapter - 5)** **[6]**

Q.7 a) *A box contains 3 white, 4 red and 5 black balls. A ball is drawn at random find the probability that is.* **[6]**

i) Red.

ii) Not black

iii) Black and white **(Refer Q.2 of Chapter - 6)**

b) *Define PDF and CDF. Also, state the properties of CDF and PDF.* **(Refer Q.10 and Q.11 of Chapter - 6)**

c) *Given the pdf for different X values as follows.* $x = 1$, *pdf* = 0.2, $x = 2$, *pdf* = 0.1, $x = 3$, *pdf* = 0.3, $x = 4$, *pdf* = 0.3, $x = 5$, *pdf* = 0.1. *Draw the pdf and its corresponding CDF. Also plot the CDF for same.*

Ans. : Here the given data is as follow :

$$P(1) = 0.2 \qquad P(4) = 0.3$$
$$P(2) = 0.1 \qquad P(5) = 0.1$$
$$P(3) = 0.3$$

Calculation of CDF $F_X(x)$:

$$F_X(x) = P(X \le x)$$

$$\therefore \quad F_X(1) = P(X<1)+P(X=1) = 0+0.2 = 0.2$$

$$F_X(2) = P(X<2)+P(X=2)$$
$$= P(X<1)+P(X=1)+P(X=2)$$
$$= 0+0.2+0.1 = 0.3$$

$$F_X(3) = P(X<1)+P(X=1)+P(X=2)+P(X=3)$$
$$= 0+0.2+0.1+0.3 = 0.6$$

Similarly,

$$F_X(4) = 0+0.2+0.1+0.3+0.3 = 0.9$$

$$\text{and } F_X(5) = 0+0.2+0.1+0.3+0.3+0.1 = 1$$

Calculation of pdf $f_X(x)$:

$$f_X(x) = \frac{d}{dx}F_X(x)$$

Thus pdf is obtained by taking change in values of cdf at every value of x.

$$\therefore \quad f_X(1) = F_X(1)-F_X(0) = 0.2-0 = 0.2$$
$$f_X(2) = F_X(2)-F_X(1) = 0.3-0.2 = 0.1$$
$$f_X(3) = F_X(3)-F_X(2) = 0.6-0.3 = 0.3$$
$$f_X(4) = F_X(4)-F_X(3) = 0.9-0.6 = 0.3$$
$$f_X(5) = F_X(5)-F_X(4) = 1-0.9 = 0.1$$

Fig. 7 shows the plots of pdf and cdf as calculated above.

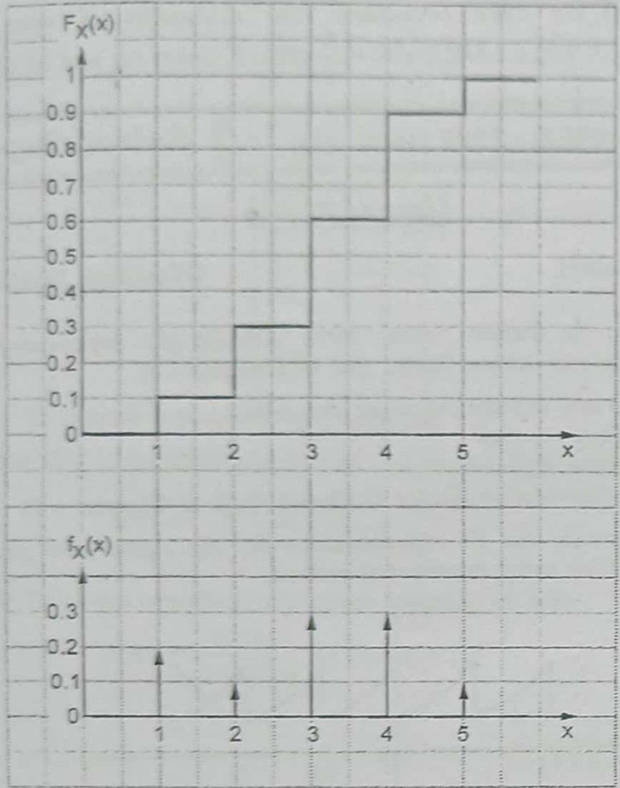


Fig. 7 : Plots of cdf and pdf

OR

Q.8 a) *What are statistical properties of random variables. State them (any 3).* (Refer Q.18 and Q.19 of Chapter - 6) [6]

b) *Two fair, six-sided dice are thrown. Find the probability of.* [5]

i) Throwing a sum of 11.

ii) Throwing two 7s.

iii) Throwing a pair

Ans. : Each dice has numbers from 1 to 6. When two dice are thrown, the sample space will be as shown below :

$$S = \begin{Bmatrix} 1,1 & 1,2 & 1,3 & 1,4 & 1,5 & 1,6 \\ 2,1 & 2,2 & 2,3 & 2,4 & 2,5 & 2,6 \\ 3,1 & 3,2 & 3,3 & 3,4 & 3,5 & 3,6 \\ 4,1 & 4,2 & 4,3 & 4,4 & 4,5 & 4,6 \\ 5,1 & 5,2 & 5,3 & 5,4 & 5,5 & 5,6 \\ 6,1 & 6,2 & 6,3 & 6,4 & 6,5 & 6,6 \end{Bmatrix}$$

Thus there will be total 36 entries in sample space.

i) The sample space for sum of 11 will be,

$$\text{sum of } 11 = \{5,6 \quad 6,5\} \Rightarrow 2 \text{ entries}$$

$$\therefore \; P \text{ (Throwing a sum of 11)} = \frac{2}{36} = \frac{1}{18}$$

ii) Maximum number on the dice is 6. Hence throwing 7 is not possible.

$$\therefore \; P \text{ (Throwing two } 7s) = \frac{0}{36} = 0$$

iii) The sample space for throwing pair will be,

$$\text{Pair} = \{1,1 \quad 2,2 \quad 3,3 \quad 4,4 \quad 5,5 \quad 6,6\} \Rightarrow 6 \text{ entries}$$

$$\therefore P \text{ (Throwing a pair)} = \frac{6}{36} = \frac{1}{6}$$

c) *Consider a fair die, plot a CDF v/s 'x' find the CDF of each value of x plot PDF and CDF.* [6]

Ans. : For tossing a dice experiment, the sample space is,

$$S = \{1, 2, 3, 4, 5, 6\}$$

To obtain CDF

$$P(X<1) = 0$$

$$P(X=1) = \frac{1}{6}$$

$$P(X=2) = \frac{1}{6}$$

$$P(X=3) = \frac{1}{6}$$

$$P(X=4) = \frac{1}{6}$$

$$P(X=5) = \frac{1}{6}$$

$$P(X=6) = \frac{1}{6}$$

$$\therefore \quad P(X<1) = 0$$

$$P(X \le 1) = \frac{1}{6}$$

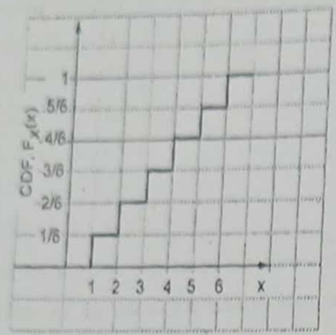


Fig. 8 : CDF of tossing a dice

$$P(X \le 2) = P(X \le 1) + P(X=2) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$$

$$P(X \le 3) = P(X \le 2) + P(X=3) = \frac{2}{6} + \frac{1}{6} = \frac{3}{6}$$

Similarly,

$$P(X \le 4) = \frac{4}{6}$$

$$P(X \le 5) = \frac{5}{6}$$

$$P(X \le 6) = 1$$

Fig. 9 shows the CDF.

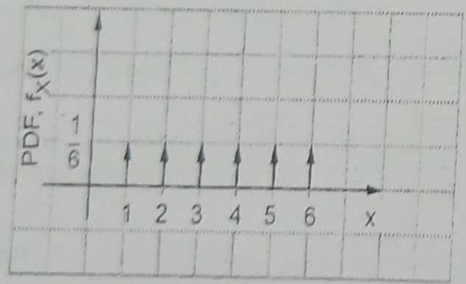


Fig. 9 : PDF of tossing a dice

To obtain PDF For each occurrence, the probability is same. $P(X=1) = P(X=2) = \ldots P(X=6) = \frac{1}{6}$. Fig. 9 shows the plot of PD

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Course 2019

Time : $2\frac{1}{2}$ Hours] **[Maximum Marks**

Q.1 a) *What is fourier series. Write formula for exponential trigonometric fourier series.* **(Refer Q.5 of Chapter - 3)**

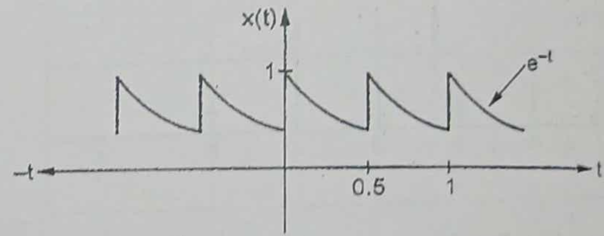
b) *State and explain following properties.*

i) Time reversal

ii) Time differentiation

iii) Convolution **(Refer Q.18 and Q.22 of Chapter - 3)**

c) *Determine the FS representation for the signal with per wave, shown below using exponential method.*



$x(t) = e^{-t}$

Fig. 1

Ans. : Step 1 : To obtain $X(k)$

$$X(k) = \frac{1}{T} \int_{<T>} x(t) e^{-jk\omega_0 t} \, dt$$

Here $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{0.5} = 4\pi$ and $x(t) = e^{-t}$ for 0 to 0.5 i.e.,

$$X(k) = \frac{1}{0.5} \int_0^{0.5} e^{-t} e^{-jk4\pi t} \, dt = 2\int_0^{t} e^{-(1+j4\pi k)t} \, dt$$

$$= 2\frac{1}{-(1+j4\pi k)}\left[e^{-(1+j4\pi k)t}\right]_0^{0.5}$$

$$= -\frac{2}{1+j4\pi k}\left[e^{-(1+j4\pi k)0.5} - e^0\right]$$

$$= -\frac{2}{1+j4\pi k}\left[e^{-0.5}\cdot e^{-j2\pi k} - 1\right]$$

$e^{-j2\pi k} = \cos 2\pi k - j\sin 2\pi k = 1$ always. Hence,

$$X(k) = -\frac{2}{1+j4\pi k}[0.606-1]$$

$$= \frac{0.7869}{1+j4\pi k} \qquad \dots (1)$$

...ep 2 : To express exponential fourier series.

...ting for $X(k)$ in synthesis equation of equation (Q.5.3) in Chapter - 3,

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{0.7869}{1+j4\pi k} e^{jk\omega_0 t}$$

...ep 3 : To obtain magnitude and phase spectrum of $X(k)$.

...e have

$$X(k) = \frac{0.7869}{1+j4\pi k} \text{ by equation (1)}$$

$$= \frac{0.7869}{1+j4\pi k} \times \frac{1-j4\pi k}{1-j4\pi k} = \frac{0.7869(1-j4\pi k)}{1+(4\pi k)^2}$$

$$= \frac{0.7869}{1+(4\pi k)^2} - j\frac{0.7869\times 4\pi k}{1+(4\pi k)^2} \qquad \dots (2)$$

$$|X(k)| = \sqrt{\frac{(0.7869)^2}{[1+(4\pi k)^2]^2} + \frac{(0.7869\times 4\pi k)^2}{[1+(4\pi k)^2]^2}}$$

$$= \sqrt{\frac{(0.7869)^2 + (0.7869)^2 (4\pi k)^2}{[1+(4\pi k)^2]^2}}$$

$$= \sqrt{\frac{(0.7869)^2 (1+(4\pi k)^2)}{[1+(4\pi k)^2]^2}}$$

$$\therefore \quad |X(k)| = \frac{0.7869}{\sqrt{1+(4\pi k)^2}}$$

And phase spectrum is given as,

$$\angle X(k) = \tan^{-1}\left[\frac{\text{Imaginary part of eq (2)}}{\text{Real part of eq (2)}}\right]$$

$$\therefore \quad \angle X(k) = -\tan^{-1}(4\pi k)$$

Following table lists the calculation of $(X(k))$ and $\angle X(k)$.

k	$\lvert X(k)\rvert = \frac{0.7869}{\sqrt{1+(4\pi k)^2}}$	$\angle X(k) = -\tan^{-1}(4\pi k)$ (in radians)
−3	0.0208	1.5442
−2	0.0312	1.5310
−1	0.0624	1.491
0	0.7869	0
1	0.0624	−1.491
2	0.0312	−1.5310
3	0.0208	−1.5442

Fig. 1(a) shows the magnitude and phase plot as per above calculations. (Refer Fig. 1(a) on next page)

Comments on Results

i) Observe that $|X(k)| = |X(-k)|$ i.e. magnitude response is symmetric since $x(t)$ is real.

ii) The phase spectrum is antisymmetric for real signal.

iii) We know that $\omega_0 = 4\pi$. Hence each value of 'k' changes the scale on x-axis by '$k\omega_0$'. i.e.

'$k\omega_0$' $= 0, \pm\omega_0, \pm 2\omega_0, \pm 3\omega_0, \dots = 4\pi k,$
$k = 0, \pm 1, \pm 2, \dots = 0, \pm 4\pi, \pm 8\pi, \pm 12\pi, \dots$

iv) The frequencies $\pm 2\omega_0, \pm 3\omega_0, \pm 4\omega_0 \dots$ are called harmonies of fundamental frequency ω_0.

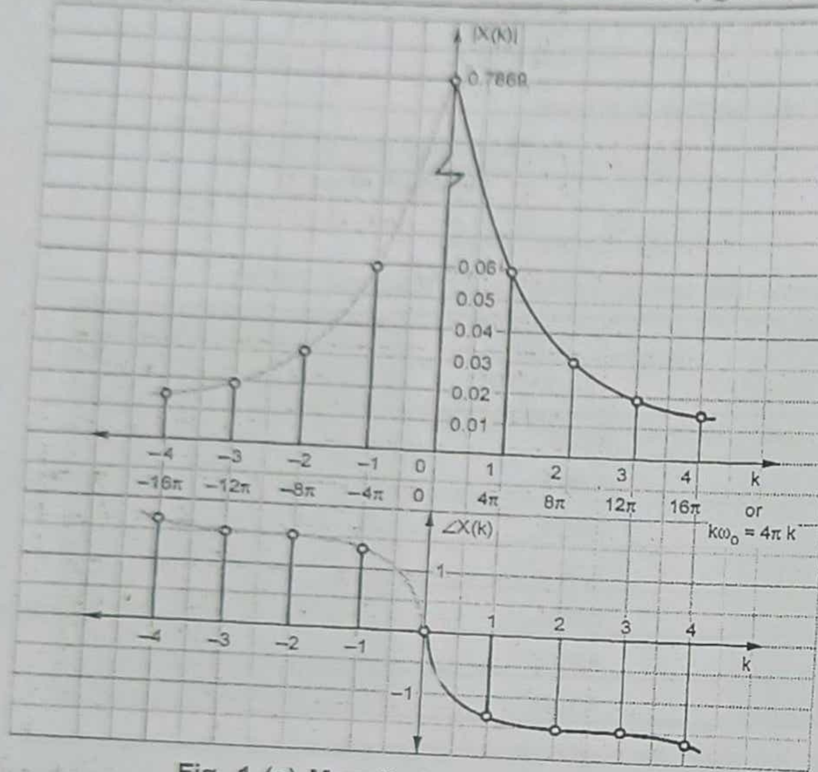


Fig. 1 (a) Magnitude and phase plot

OR

Q.2 a) *Find the trigonometric Fourier series for the periodic signal x(t) given below.* (Refer Q.12 of Chapter - 3) [8]

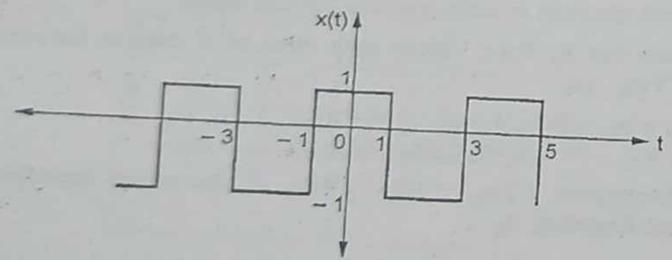


Fig. 2

b) *State the following properties of CTFS.*
i) Time scaling ii) Time Integration iii) Modulation

Ans. : Scaling :

$$z(t) = x(at) \xleftrightarrow{FT} Z(k) = X(k)$$

Proof :
$$X(k) = \frac{1}{T}\int_{<T>} x(t)e^{-jk\omega_0 t}\,dt$$

- Since $x(t)$ is periodic, then $z(t) = x(at)$ is also periodic. And if T the period of $x(t)$, then period of $z(t)$ will be $\frac{T}{a}$.
- Similarly if frequency of $x(t)$ is ω_0, the frequency of $z(t) = x(at)$ be $a\omega_0$, since 't' is multiplied by factor 'a'.

Therefore Fourier coefficients of $z(t)$ can be written as,

$$Z(t) = \frac{1}{\left(\frac{T}{a}\right)}\int_{\left(\frac{T}{a}\right)} z(t)e^{-jk(a\omega_0)t}\,dt$$

$$= \frac{a}{T}\int_{\left(\frac{T}{a}\right)} x(at)e^{-jka\omega_0 t}\,dt$$

Put $at = m$, then $dt = \frac{1}{a}dm$, then above equation becomes,

$$Z(k) = \frac{a}{T}\int_{\left(\frac{T}{a}\right)} x(m)e^{-jk\omega_0 m}\cdot\frac{1}{a}\,dm$$

$$= \frac{1}{T}\int_{\left(\frac{T}{a}\right)} x(m)e^{-jk\omega_0 t}\,dm = X(k)$$

Comment : Fourier coefficients of $x(t)$ and $x(at)$ are same, but spacing between frequency components change from ω_0 to $a\omega_0$.

Multiplication or Modulation Theorem :

$$z(t) = x(t)\,y(t) \xleftrightarrow{FS} Z(k) = X(k) * Y(k)$$

Proof :

$$Z(k) = \frac{1}{T}\int_{<T>} z(t)e^{-jk\omega_0 t}\,dt$$

$$= \frac{1}{T}\int_{<T>} [x(t)\,y(t)]e^{-jk\omega_0 t}\,dt \text{ putting for } z(t)$$

By synthesis equation, $x(t) = \sum_{k=-\infty}^{\infty} X(k)e^{jk\omega_0 t}$. Putting this expression for $x(t)$ in above equation,

$$Z(k) = \frac{1}{T}\int_{<T>} \sum_{k=-\infty}^{\infty} X(m)e^{jm\omega_0 t}\cdot y(t)e^{-jk\omega_0 t}\,dt$$

Note that index of summation is changed in above equation to differentiate between two indices of 'k' and 'm'. Interchanging the order of integration and summation,

$$Z(k) = \sum_{k=-\infty}^{\infty} X(m)\left[\frac{1}{T}\int_{<T>} y(t)e^{-j(k-m)\omega_0 t}\,dt\right]$$

The quantity inside the bracket indicates Fourier coefficients $Y(k-m)$. Hence above equation will be,

$$Z(k) = \sum_{k=-\infty}^{\infty} X(m)Y(k-m)$$

i.e.

$$Z(k) = X(k) * Y(k)$$

c) *Explain Gibb's phenomenon of Fourier series.* **(Refer Q.9 of Chapter - 3)** [4]

Q.3 a) *Find the Inverse Fourier Transform using partial fraction expansion.*

$$X(j\omega) = \frac{1}{(j\omega)^2 + 5j\omega + 6}$$ [7]

Ans. : The given function can be written as,

$$X(j\omega) = \frac{1}{(j\omega+2)(j\omega+3)}$$

$$= \frac{1}{j\omega+2} - \frac{1}{j\omega+3}$$

Taking inverse fourier transform

$$x(t) = e^{-2t}u(t) - e^{-3t}u(t)$$

b) *Find the Fourier Transform of a constant signal AO.* **(Refer Q.11 (i) of Chapter - 4)** [6]

c) *Find the Fourier Transform of a*
i) $x(t) = \delta(t) + u(t)$
ii) $x(t) = u(-t)$
Using properties of F.T. [4]

Ans. : i) $x(t) = \delta(t) + u(t)$

$$X(\omega) = FT\,\{\delta(t) + u(t)\}$$

By linearity property,

$$X(\omega) = FT\,\{\delta(t)\} + FT\,\{u(t)\}$$

$$= 1 + \frac{1}{j\omega}$$

ii) $x(t) = u(-t)$

We know, $u(t) = \frac{1}{j\omega}$

By time reversal property, $u(-t) = -\frac{1}{j\omega}$

OR

Q.4 a) *State any six properties of fourier transform.* **(Refer Q.8, Q.9 and Q.10 of Chapter - 4)** [6]

b) *Find the Fourier Transform of the signum function.* **(Refer Q.11 (ii) of Chapter - 4)** [7]

c) *Obtain the Inverse Fourier Transform of*

$$X(j\omega) = \frac{2}{j\omega+1} + \frac{1}{j\omega+2}$$ [4]

Ans. : Here use the standard FT pair,

$$e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{a+j\omega}$$

Taking inverse fourier transform,

$$x(t) = 2e^{-t}u(t)+e^{-2t}u(t)$$

Q.5 a) *Find the Laplace Transform and find ROC.*
$x(t) = e^{-3t}u(t)+e^{-2t}u(t)$ **(Refer Q.14 of Chapter - 5)** **[6]**

b) *State and explain Initial value theorem and final value theorem.* **(Refer Q.21 and Q.22 of Chapter - 5)** **[6]**

c) *Find the Inverse Laplace Transform of* $X(s) = \frac{2}{(s+4)(s-1)}$ *if the ROC is* $-4 \le R_e(s) < 1$. **(Refer Q.32 of Chapter - 5)** **[6]**

OR

Q.6 a) *Find the Laplace Transform of the signal drawn below Find ROC.* **[6]**

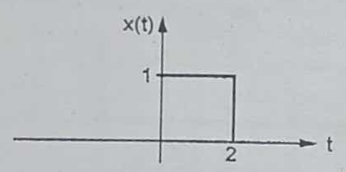


Fig. 3

Ans. : The given square pulse can be expressed as,

$$x(t) = u(t)-u(t-2)$$

Let us use standard LT pair, $u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s}$ and time shift property. Then Laplace Transform of above equation becomes,

$$X(s) = \frac{1}{s}-\frac{1}{s}e^{-2s}$$

$$= \frac{1-e^{-2s}}{s}, \; ROC : s > 0$$

b) *Solve the differential equation* $\frac{dy(t)}{dt}+3y(t) = x(t)$ *for input* $x(t) = e^{-2t}u(t)$. *Assume zero initial conditions.*
(Refer Q.35 of Chapter - 5) **[6]**

c) *Find the Laplace Transform of following using the properties.*
i) $x(t) = \frac{d}{dt}u(t)$
ii) $x(t) = u(t+1)$ **[6]**

Ans. : i) $x(t) = \frac{d}{dt}u(t)$

We have $u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s}$

and time differentiation property states,

$$\frac{d}{dt}x(t) \xleftrightarrow{\mathcal{L}} sX(s)$$

Hence $\quad \mathcal{L}\{x(t)\} = \mathcal{L}\left\{\frac{d}{dt}u(t)\right\}$

$$= s \cdot \mathcal{L}\{u(t)\} = s \cdot \frac{1}{s} = 1$$

ii) $x(t) = u(t+1)$

We have $u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s}$

And time shift property states,

$$x(t-t_0) \xleftrightarrow{\mathcal{L}} e^{-st_0}X(s)$$

Hence $\quad \mathcal{L}\{x(t)\} = \mathcal{L}\{u(t+1)\}$

$$= e^{s}\mathcal{L}\{u(t)\}$$

$$= e^{s} \cdot \frac{1}{s} = \frac{e^{s}}{s}$$

Q.7 a) *Define the following terms :*
i) Probability ii) Joint probability iii) Conditional probability.
(Refer important points to Remeber of section 6.1) **[6]**

b) *A coin is tossed three times. Write the sample space which gives all possible out comes. A random variable X. Which represents the number of heads obtained on any tipple toss. Calculate and draw the CDF and PDF.* **(Refer Q.13 of Chapter - 6)** **[7]**

c) *In a pack of cards, 2 cards are drawn simultaneously. What is the probability of getting a Queen and Jack combination ?*
(Refer Q.3 of Chapter - 6) **[4]**

OR

Q.8 a) *Define probability. Also write the properties of probability.*
(Refer Important points to remember of section 6.1 and Q.1 of Chapter - 6) **[5]**

b) *A perfect die is thrown. Find the probability that i) You get even number ii) You get perfect square.*
(Refer Q.4 of Chapter - 6) **[6]**

c) *The probability density function of a random variable 'X' is given by*

$$f_x(x) = \frac{1}{a} \qquad |x| \leq a$$
$$= 0 \qquad \text{otherwise}$$

Determine : i) Mean, ii) Mean square iii) Standard deviation.
(Refer Q.23 of Chapter - 6) **[6]**

END...